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FUNCTIONS OF LIMITED VARIATION AND LEBESGUE INTEGRALS.

By Goldie Printis Horton.

1. **Introduction.** A fundamental theorem in the theory of Lebesgue integrals is that the four derivates of a function continuous and of limited variation are summable and all equal except over a set of measure zero. This theorem is proved by Lebesgue* by actually calculating the variation of the function and by Vallée Poussin† by using two auxiliary functions, called majorating and minorating functions, which, with their derivatives, satisfy certain conditions of inequality.

The purpose of this paper is to show how the proof of this theorem can be simplified by the study of a very simple monotone function, which we shall name *measure function*, preliminary to proving directly Lebesgue's theorem that a continuous function with a bounded derivate has a derivative except for a set of measure zero, and to the deduction of known existence theorems for derivatives independently of the theory of majorating functions.

2. **Definitions and known theorems.** For the convenience of the reader we state the following definitions:

Definition I. Given a set of points E bounded and contained in an interval (a, b). Enclose the points of E in a set of intervals. Let $\Sigma\alpha$ denote their sum. By definition, the exterior measure of E, denoted by m_eE, is the lower limit of all the sums $\Sigma\alpha$ possible; the interior measure of E, denoted by m_iE, is $(b - a) - m_eCE$;‡ and in case $m_eE = m_iE$, the set E is said to be measurable, and its measure, mE, is the common value of m_eE and m_iE.§

Definition II. A function $f(x)$, one-valued in a measurable set E and of determinate sign if infinite, is said to be measurable in E if at least one of the sub-sets of E for which $f \geqq A, f < A, f > A, f \leqq A$ is measurable whatever constant A may be.‖ It follows then that the other three are also measurable.

* Leçons sur l'Integration, p. 121.

† Cours d'Analyse, vol. 1, 3d ed., § 261.

‡ CE denotes the complement of E with respect to (a, b), that is, the points of (a, b) not in E.

§ If $mE = 0$, E is called a *nul* set or a set of measure zero.

‖ Of the properties of measurable functions attention is called to the facts that the limit or the limits of indetermination of a sequence of measurable functions is measurable, that any continuous function is measurable, and hence the derivates (the right- and left-hand limits of indetermination of the incremental ratio) of a continuous function are measurable. For proofs

DEFINITION III. Let the function $f(x)$ be single-valued, measurable, and with bounds μ and M in a measurable set E. Divide its interval of variation (μ, M) into consecutive parts by the series of increasing numbers $l_1 = \mu, l_2, l_3, \cdots, l_{n+1} = M$, and let η_i be a value of $f(x)$ anywhere on the interval (l_i, l_{i+1}). Let e_i denote the measure of the sub-set e_i of E where $l_i \leqq f(x) < l_{i+1}$. It can be proved that $\lim\limits_{n=\infty} \sum\limits_{i=1}^{n} \eta_i e_i$ exists, and by definition this limit is the Lebesgue integral of $f(x)$ in the set E and is written $L_E f(x)dx$, or $\int_E f(x)dx$ in case there is no confusion with the Riemann integral.

In case $f(x)$ is not bounded, and not negative in E, let $f_n = f(x)$ at points of E where $f(x) \leqq n$, and $f_n = n$ where $f(x) > n$. Then by definition $\int_E f(x)dx = \lim\limits_{n=\infty} \int_E f_n dx$, and if this limit is finite $f(x)$ is said to be *summable* in E.

In the general case, a non-bounded function $f(x)$ is the difference of two non-negative functions, and $f(x)$ is *summable* if these two functions are summable.*

DEFINITION IV. Let $f(x)$ be bounded in an interval (a, b), $a < b$. Let (α, β) be a sub-interval of (a, b) such that $a \leqq \alpha < \beta \leqq b$, and call $f(\beta) - f(\alpha)$ the variation of $f(x)$ in (α, β). Consider a set (α_i, β_i) of distinct (α, β) intervals. If for every δ, however small, an ϵ can be found such that for every set (α_i, β_i) of intervals $\Sigma[f(\beta_i) - f(\alpha_i)] < \delta$ when $\Sigma(\alpha_i, \beta_i) < \epsilon$, $f(x)$ is said to be *absolutely continuous* in (a, b).†

Reference will be made to the following known theorems:

(A). THEOREM ON CONVERGENCE. If a sequence $f_1, f_2, \cdots, f_n, \cdots$ of measurable functions converge toward a finite limit $f(x)$ in a measurable set E, for every ϵ and δ, however small, a value N of n can be found such that $|f(x) - f_n(x)| < \epsilon$ except in a set of measure $< \delta$ for all values of $n > N$.‡

of these properties see Lebesgue's Leçons sur l'Intégration, Vallée Poussin's Cours d'Analyse, Vallée Poussin's Intégrales de Lebesgue, Bliss's Integrals of Lebesgue, Bull. Amer. Math. Soc., vol. 24 (1917), pp. 1–46.

* It is to be noted that a set of measure zero can be neglected in the calculation of an integral, and that it then follows from the definition that a function summable in a set E can become infinite in a sub-set of E of measure zero. It can be proved that a function measurable and bounded is summable; and hence if one derivate of a continuous function is bounded all the derivates are bounded and summable. It follows from the definition that if a function is summable its absolute value is summable and inversely.

† Vitali, "Sulle funzioni integrali," in Atti della R. Accademia delle Scienze di Torino, 1905. The statement given here is that of M. B. Porter in "Concerning Absolutely Continuous Functions," Bull. Am. Math. Soc., vol. 22 (1915), pp. 109–111.

‡ For a proof of this theorem see Vallée Poussin, Cours d'Analyse, p. 71.

(B). Distributive Properties of Summable Functions.

$$\sum_1^\infty \int_{E_i} f(x)dx = \int_{\sum_1^\infty E_i} f(x)dx$$

if the E_i have no points in common, and $\sum_1^n \int_E f_i(x)dx = \int_E \sum_1^n f_i(x)dx$ if the f_i are finite in number.*

(C). Theorem of the Mean. If $f(x)$ is summable in a set E, and $\mu \leqq f(x) \leqq M$, μ and M finite, μ meas $E \leqq \int_E f(x)dx \leqq M$ meas E.†

(D). Lebesgue's Limit Theorem. If $\lim_{n=\infty} f_n(x) = f(x)$ and $f_n(x)$ is bounded and measurable in E, $\lim_{n=\infty} \int_E f_n(x)dx = \int_E f(x)dx$.‡

(E). Theorem. A function continuous in an interval cannot have a derivate as Λ negative (positive) only in a set of measure zero and have that derivate finite in that set.§

For later reference we prove the following

Lemma. *If* $f(x)$ *is continuous in* (a, b)

$$f(x) - f(a) \equiv V_a^x f(x) = \lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx, \; x < b,$$

uniformly for all values of x *in* (a, b), *where* $\int$ *denotes the Riemann integral.*‖

Consider the identity

$$\lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx$$

$$\equiv \lim_{h=0} \left[\frac{\int_a^{x+h} f(x)dx - \int_a^x f(x)dx}{h} - \frac{\int_a^{a+h} f(x)dx - \int_a^a f(x)dx}{h} \right]$$

$$\equiv \lim_{h=0} \left[\frac{\int_x^{x+h} f(x)dx}{h} - \frac{\int_a^{a+h} f(x)dx}{h} \right].$$

* Ibid., § 249.

† Ibid., § 246.

‡ Ibid., § 250.

§ The upper and lower right-hand derivates of a function $F(x)$ are denoted by ΛF and λF respectively. Vallée Poussin proves (Cours d'Analyse, § 115) by means of elementary properties of derivates that if the set E where a derivate of a continuous function has the same sign is of measure zero, the derivate becomes infinite in a part of E. But more follows from Vallée Poussin's proof than his statement implies, and it is this stronger result that is stated in (E).

‖ See Lebesgue, Leçons sur l'Intégration, p. 120.

Applying the mean value theorem to the integrals on the right, we have

$$\lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx = \lim_{h=0} [f(x + \vartheta_1 h) - f(a + \vartheta_2 h],$$

where $0 < \vartheta_1 < 1$, $0 < \vartheta_2 < 1$ in (a, b). Since $f(x)$ is continuous,

$$\lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx = f(x) - f(a),$$

uniformly.

2. **The function** $M(x)$. Consider a function, which we shall name the *measure function*, defined as follows:

Let δ_i be any set of non-overlapping intervals (and therefore a denumerable set of intervals) in the interval (a, b) and let $M(x)$ denote the sum of the intervals δ_i or parts thereof to the left of x. We may write $M(x) = \sum_a^x \delta_i$. Important properties of $M(x)$ are included in the following

THEOREM. *The function $M(x)$, defined for all real values of x, is absolutely continuous, monotone increasing, and always less than or equal to $\sum_1^\infty \delta_i$. It has a derivative $M'(x)$ for all x's except for a nul set.*

To show that $M(x)$ is absolutely continuous it need be noted merely that its variation over any set of intervals η_i is $\leqq \Sigma\eta_i$. It is evident that $M(x) \leqq \sum_1^\infty \delta_i$, and also that it is monotone increasing, that is, ΛM and λM are always positive or zero.

Denoting by Δ the set of points covered by the δ's and by $C\Delta$ the set complementary to Δ with respect to (a, b), it is evident that $M(x)$ has a derivative $M'(x) = 1$ for inner points of Δ. We shall show that $M'(x) = 0$ for all points of $C\Delta$ except for a set of measure zero.

By the preceding lemma and (B),

$$M(b) = \lim_{h=0} \int_\Delta \frac{M(t+h) - M(t)}{h} dt + \lim_{h=0} \int_{C\Delta} \frac{M(t+h) - M(t)}{h} dt,$$

from which we have

$$\lim_{h=0} \int_{C\Delta} \frac{M(t+h) - M(t)}{h} dt = 0,$$

since $M'(x) = 1$ for any inner point of the δ's. Since ΛM and λM are always positive or zero, this shows that $\lambda M = 0$ in $C\Delta$ except for a nul set, and therefore $M(x) = \int_a^x \lambda M(t) dt$.

Similarly,

$$\begin{aligned}
\text{meas } C\Delta = \int_a^b \lambda(t - M(t))dt = \int_a^b (1 - \Lambda M)dt \\
&= \int_\Delta (1 - \Lambda M)dt + \int_{C\Delta} (1 - \Lambda M)dt \\
&= \int_{C\Delta} (1 - \Lambda M)dt \\
&= \text{meas } C\Delta - \int_{C\Delta} \Lambda M dt.
\end{aligned}$$

Thus $\int_{C\Delta} \Lambda M dt = 0$. Hence $\lambda M = \Lambda M = 0$ in $C\Delta$ except for a nul set, which completes the theorem.

This theorem is a special case of Lebesgue's general theorem that any continuous function of limited variation in an interval has a derivative at all points of that interval except for a nul set, but the proof of the theorem for this measure function is much simpler than that for the general function of limited variation. It is upon the proof for the special case that we shall base a proof for the general case.

3. Theorem I. *A function $f(x)$ continuous in an interval and with a bounded derivate has a derivative almost everywhere in that interval.**

Let K be the upper bound of $|\Lambda f|$ in the interval, and note that if one derivate of $f(x)$ is bounded all of them are bounded.† From the Theorem on Convergence (A), it follows that if ϵ and δ are positive and arbitrarily small, writing $\Delta f = f(x + h) - f(x)$, $\Lambda f + \epsilon > \Delta f/h$ except for a set E_1 of measure $< \delta$ and $\Delta f/h + \epsilon > \lambda f$ except for a set E_2 of measure $< \delta$, if h is small enough. Now enclose E_1 and E_2 in sets of open intervals δ_i and δ_i', form the two measure functions $M_{E_1}(x)$ and $M_{E_2}(x)$‡ and write

$$\varphi_1(x) = KM_{E_1}(x), \qquad \varphi_2(x) = KM_{E_2}(x).$$

Then the functions

$$\Psi_1(x) = f(x) + \epsilon x + \varphi_1(x) - \int_a^x \frac{\Delta f}{h} dx$$

and

$$\Psi_2(x) = \int_a^x \frac{\Delta f}{h} dx + \epsilon x + \varphi_2(x) - f(x)$$

* "Almost everywhere in an interval" means "except for a set of measure zero in that interval."

† This fact is due to Dini. See Vallée Poussin, Cours d'Analyse, § 112, II.

‡ These functions are determinate only when the sets of intervals δ_i and δ_i', have been chosen.

are each always monotone increasing, since

$$\Lambda\Psi_1(x) = \Lambda f + \epsilon + \varphi_1' - \frac{\Delta f}{h} > 0,$$

$$\Lambda\Psi_2(x) = \frac{\Delta f}{h} + \epsilon + \varphi_2' - \lambda f > 0.$$

So that

$$\lambda\Psi_1(x) = \lambda f + \epsilon + \varphi_1' - \frac{\Delta f}{h} > 0,$$

$$\lambda\Psi_2(x) = \frac{\Delta f}{h} + \epsilon + \varphi_2' - \Lambda f > 0.$$

Hence

$$\frac{\Delta f}{h} - \varphi_1' - \epsilon < \Lambda f < \frac{\Delta f}{h} + \varphi_2' + \epsilon,$$

$$\frac{\Delta f}{h} - \varphi_1' - \epsilon < \lambda f < \frac{\Delta f}{h} + \varphi_2' + \epsilon.$$

Thus, except for a set $E_1 + E_2$ of measure $< 2\delta$, Λf and λf differ from $\Delta f/h$ by a quantity $\leqq 2\epsilon$, which shows that $\Lambda f = \lambda f$ except for a nul set, which proves the theorem.

THEOREM II. *The indefinite integral of a function measurable and bounded has that function as a derivative almost everywhere.*

Let $f(x)$ be measurable and bounded in (a, x), and let

$$F(x) - F(a) = \int_a^x f(x)dx. \tag{1}$$

It is to be noted that $F(x)$ is absolutely continuous and hence measurable.

Since $f(x)$ is bounded, the Theorem of the Mean (C) is applicable and we have

$$\min f(x) \leqq \frac{F(x+h) - F(x)}{h} = \int_x^{x+h} \frac{f(x)}{h} dx \leqq \max f(x).$$

That is, ΛF is bounded, and therefore, by Theorem I, $F'(x)$ exists almost everywhere.

According to the preceding lemma

$$F(x) - F(a) = \lim_{h=0} \int_a^x \frac{F(x+h) - F(x)}{h} dx,$$

the Riemann integral of the lemma being replaced by a Lebesgue integral since the integrand is bounded and measurable. Then by Lebesgue's Theorem (D) relative to the passage to the limit under the integral sign, and by Theorem I,

$$F(x) - F(a) = \int_a^x F'(x)dx. \tag{2}$$

From (1) and (2) it follows that $\int_a^x [F'(x) - f(x)]dx = 0$, and also $\int_i [F'(x) - f(x)]dx = 0$, where i is any interval or finite set of intervals in (a, x). Then $F'(x) = f(x)$ almost everywhere, for suppose $F'(x) - f(x) > k$, k positive, over a set E of measure δ. Enclose E in a finite number of intervals α_i the sum of whose lengths differ from δ by as little as you please. Then $\int_i [F'(x) - f(x)]dx > k\delta$, which is a contradiction. The supposition $f(x) - F'(x) > k$ leads to a contradiction simiarly, and the theorem is proved.

The truncation process of Vallée Poussin can be used to deal with the case where the integrand is not bounded. We prove

THEOREM III. *If a continuous function $F(x)$ is non-decreasing in an interval (a, x), any one, as Λ, of its derivates is summable and*

$$\int_a^x \Lambda F dx \leqq F(x) - F(a).$$

Let E be the set of points where $\Lambda F \leqq n$, a positive number, and let CE denote the set complementary to E with respect to the interval (a, x). Let $\Lambda_n = \Lambda F$ in E, and $\Lambda_n = n$ in CE. Then Λ_n is bounded and measurable in (a, x) and hence summable. Let $\Phi_n(x) = \int_a^x \Lambda_n dx$. By Theorem II, $\Phi_n'(x) = \Lambda_n$ almost everywhere. Then $\Phi_n'(x) = \Lambda F$ except for a nul set of E and the set CE. By the Theorem of the Mean (C), for points of CE,

$$n \leqq \frac{\Phi_n(x + h) - \Phi_n(x)}{h} = \frac{\int_x^{x+h} \Lambda_n dx}{h} \leqq \Lambda F,$$

that is, $\Lambda F - \Phi_n'(x) \geqq 0$ everywhere in CE. Then $\Lambda F - \Phi_n'(x) < 0$ only in a nul set of E. But in E this difference is always bounded, and hence by Theorem (E) it is never negative, and $\Lambda F \geqq \Phi_n'$ always. That is

$$\int_a^x \Lambda_n dx \leqq F(x) - F(a).$$

Let n become infinite. Since the left-hand member is bounded by the right-hand member, it approaches a finite limit, which by Definition II is $\int_a^x \Lambda F dx$. That is,

$$\int_a^x \Lambda F dx \leqq F(x) - F(a).$$

4. FUNDAMENTAL THEOREM IV. *A function $F(x)$ continuous and of limited variation in an interval (a, x) has any one of its derivates, as Λ, summable in that interval.*

Since $F(x)$ is continuous and of limited variation $F(x) = F_1(x) - F_2(x)$ where $F_1(x)$ and $F_2(x)$ are continuous and non-decreasing. Since $\Lambda F = \Lambda(F_1 - F_2) \geqq \Lambda F_1 - \Lambda F_2$, and $\Lambda(F_1 + F_2) \leqq \Lambda F_1 + \Lambda F_2$, we may write

$$\Lambda F_1 + \Lambda F_2 \geqq \Lambda(F_1 + F_2) \geqq \Lambda(F_1 - F_2) \geqq \Lambda F_1 - \Lambda F_2.$$

The derivates ΛF_1 and ΛF_2 are summable (Theorem III) and, in view of the remarks on Definitions II and III, it follows that $\Lambda F = \Lambda(F_1 - F_2)$ is summable.

5. In conclusion, we prove

THEOREM V. *The integral of the derivate of a function absolutely continuous in an interval is the variation of the function in that interval.*

Let $f(x)$ be a function absolutely continuous in the interval (a, x). Since $f(x)$ is absolutely continuous, it is of limited variation. Then Λf is summable (Theorem IV) and hence the measure of the set E of points where $\Lambda f \geqq M$, M a number large at pleasure, is small with $1/M$. Let the points E be enclosed in a set of non-overlapping intervals δ_i where $\Sigma\delta_i$ is small with $1/M$. Let $C\delta$ denote the set complementary to $\Sigma\delta_i$ with respect to (a, x).

By the lemma above

$$f(x) - f(a) = \lim_{h=0} \int_a^x \frac{f(t+h) - f(t)}{h}\,dt$$

$$= \lim_{h=0} \int_{C\delta} \frac{f(t+h) - f(t)}{h}\,dt + \lim_{h=0} \int_{\Sigma\delta_i} \frac{f(t+h) - f(t)}{h}\,dt.$$

Since over $C\delta$ we have

$$\left| \frac{f(t+h) - f(t)}{h} \right| < M,$$

the first limit in the last member of the preceding equation is found by (D) to be $\int_{C\delta} f'(t)dt$. The last integral equals $\Sigma_\delta[f(x_{i+1}) - f(x_i)]$ which, since $f(x)$ is absolutely continuous, approaches zero with $\Sigma\delta_i$. That is, if $f(x)$ is absolutely continuous,

$$f(x) - f(a) = \int_a^x f'(t)dt.$$

Since the integral of any summable function is absolutely continuous, we have

VITALI'S THEOREM. *A necessary and sufficient condition that a function be the indefinite integral of a derivate is that it be absolutely continuous.*

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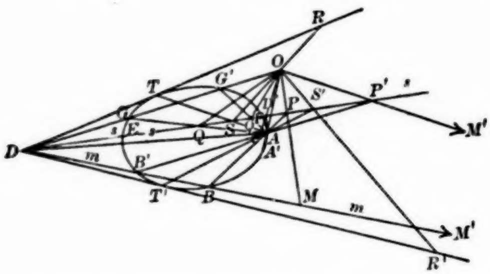
ON THE TEIXEIRA CONSTRUCTION OF THE UNICURSAL CUBIC.*

By Nathan Altshiller.

The following is a generalization and a synthetic discussion of the construction of the unicursal cubic due to Teixeira:†

1. Consider a point O, a line s, a conic (C) and two points D and A on s and (C) respectively. A variable line through A meets s in P and (C) again in B. Let $M \equiv (OP, DB)$.

The locus of M is, in general, a unicursal cubic having O for its double point, passing through D and through the points common to s and (C).



Let B' be the second point of intersection of DB with (C), and $P' \equiv (AB', s)$. The point $M' \equiv (OP', DB')$ is clearly another point of the required locus.

When the line $m \equiv DBB'$ turns about the point D describing the pencil of rays (D), the pairs of points B, B' describe an involution on (C), which involution is projected from the fixed point A by an involution of rays (A). The involution of points P, P' on s perspective to (A) is projected from the fixed point O by an involution of rays $O(PP', \cdots)$. We thus have a projective one-to-two correspondence between the pencil (D) and the involution (O). Hence the points M, M' describe a unicursal cubic (C_3) with its double point at O and passing through D.‡ The construction also shows immediately that when m coincides with s, the corresponding points of the locus are the points common to s and (C).

1, *a*. If the line s coincides with the line at infinity, the above construction is identical with Teixeira's.

* Read before the American Mathematical Society, Southwestern Section, Dec. 1, 1917.

† *Nouvelles Annales de Mathématiques*, vol. (1917), pp. 281–284.

‡ Dr. Emil Weyr. Theorie der mehrdeutigen Elementargebilde, etc., p. 13, Leipzig, Teubner, 1869.

1, *b*. It is assumed in the above that the three points O, A, D are distinct, that neither of the points O, A lies on s, nor does the point D lie on (C). Assumptions to the contrary would cause the locus to degenerate or would render the construction meaningless. They are excluded from what follows.

2. Let G, G' be the points of intersection of OD with (C). To the line DO of (D) correspond in (O) the lines projecting from O the traces Q, Q' on s of AG, AG'. Hence: *The lines OQ, OQ' are the tangents to the cubic (C_3) at the double point O.*

2, *a*. The cubic will be crunodal if the line OD cuts (C) in two real points, and acnodal if these points are conjugate imaginary. If O lies on (C) the line OA will be one of the tangents to the cubic at O.

2, *b*. The cubic will be cuspidal, if and only if OD is tangent to (C). The cuspidal tangent joins O to the trace on s of the line AT joining A to the point of contact T of OD with (C). If O coincides with T, the cuspidal tangent will be the line OA.

2, *c*. One of the tangents OQ, OQ' will coincide with OD if the point A coincides with G or G', i. e., when the points O, A, D are collinear. The line OD becomes a united element of the two forms (D) and (O), and the cubic degenerates. This case is excluded from the following considerations.

3. To the ray DA of (D) correspond in (A) the line AD and the tangent a to (C) at A. Let $D' \equiv (as)$. The point of intersection of DA with OD coincides with D, and let $C \equiv (DA, OD')$. Hence: *The line DA is the tangent to the cubic at D. The point C is the tangential of D.*

4. The double elements of the involution (A) are the rays projecting from A the points of contact T, T' of the tangents from D to (C). Let $S \equiv (s, AT)$, $S' \equiv (s, AT')$. The rays OS, OS' are the double elements of the involution (O). The two points of intersection of DT with the cubic (C_3) thus coincide with $R \equiv (DT, OS)$, and those of DT' with (C_3) in $R' \equiv (DT', OS')$. Consequently: *The tangents from D to the conic are also the tangents from D to the cubic, the points of contact with the latter being the points R, R'.*

4, *a*. If the cubic is cuspidal [2, *b*]* one of the tangents from D to (C), and therefore to (C_3), say DT, will coincide with DO, and the point R will coincide with the point O. The two curves will be tangent at T', if the points O, A, T' are collinear.

4, *b*. One of the points R, R', say R, will coincide with D, if and only if the point A coincides with T. Then D is a point of inflection and DA the inflectional tangent. If, in addition, the point O coincides with T', the line $T'A$ is the cuspidal tangent [2, *b*].

* A reference of this sort is to § 2, *b*.

5. Let A' be the second point of intersection of OA with (C). The line OA is one of the two elements of the involution (O), which correspond to the ray DA' of (D). Hence: *The second point common to OA and (C) belongs to the cubic.*

5, *a*. If O lies on (C), the point A' coincides with O, and OA is one of the tangents to the cubic at O (or the cuspidal tangent).

If OA is the tangent to (C) at A, the point A' coincides with A, i. e., A is the tangential of D.

6. We shall now consider the converse proposition. Let (C_3') be a given unicursal cubic and let O denote its double point. Let s be an *arbitrarily chosen straight line*, not passing through O, meeting the cubic in a real point D, and in a pair of points E, F, real, or conjugate imaginary, or coincident, distinct from D. Let A be an *arbitrary* point, distinct from D, on the tangent at D to the cubic. The points Q, Q' being the traces on s of the tangents to (C_3') at the double point O, let $G \equiv (AQ, OD)$, $G' \equiv (AQ', OD)$. The five points A, E, F, G, G' determine a conic (C). This conic, the line s and the points D, O, A, if made to play the same parts, as the similarly named elements in construction [1], will generate a cubic (C_3). The two curves (C_3') and (C_3) will have in common: (i) The double point O [1]; (ii) the tangents OQ, OQ' at the double point [2]; (iii) the three points D, E, F [1]; (iiii) the tangent DA at the point D [3]. Consequently the two cubics are identical.

If the cubic (C_3') has a cusp at O, the conic is to be taken tangent to the line OD at the trace T on OD of the line joining A to the point of intersection of s with the cuspidal tangent. The point T may coincide with O [2, *b*].

If the trace on s of one of the tangents at the double point is taken for the point A, the point O will take the place of one of the two points G, G' in the determination of the conic (C) [5, *a*]; and if the cubic has a cusp at O, the conic is to be taken tangent to OD at O [4, *b*].

The three points D, E, F, will coincide in D, if D is a point of inflection of the given cubic (C_3'), and if s is taken to coincide with the inflectional tangent at D. In the construction [1] the point A is necessarily a point on the tangent at D to the cubic [3], i. e., A in this case has to be a point of s, hence the given cubic cannot be generated by the above construction [1, *b*].

Consequently: *With an arbitrarily chosen straight line an infinite number of conics may be associated in order to generate a given unicursal cubic by the construction [1], provided the line does not pass through the double point of the cubic and is not an inflectional tangent.*

6, *a*. The above discussion solves the problem: *Construct a unicursal*

cubic given: (i) The double point and the two tangents at this point (or the cusp and the cuspidal tangent); (ii) a point D and the tangent at that point; (iii) two points of the cubic collinear with D (or the point of contact of one of the tangents from D to the cubic).

6, *b*. If the line s is taken to coincide with the line at infinity, the restrictions to which s is subjected preclude the possibility of generating, by construction [1], a unicursal cubic having its double point at infinity or having the line at infinity for an inflectional tangent.

7. Given the cubic (C_3') and the line s [6], the conic (C) may also be determined in the following way: Let R, R' be the points of contact of the cubic with the tangents from D to the curve, and let $S \equiv (s, OR)$, $S' \equiv (s, OR')$, $T \equiv (AS, DR)$, $T' \equiv (AS', DR')$. For (C) may be taken the conic which passes through A and is tangent to DR, DR' at the points T, T' respectively [4].

The reader may, referring to the remarks of [4], discuss the special cases, when: (*a*) O is a cusp; (*b*) D is a point of inflection; (*c*) s is one of the tangents from D to the cubic, and the possible combinations of these cases.

The above discussion solves the problem: *Construct a unicursal cubic, given the double point O, a point D, the tangent at this point, and the points of contact R, R', of the tangents from D to the cubic.* (Any line through D may be taken for s.)

8. Let A' be the point of intersection of (C_3') with OA, and C the tangential of D. Let $D' \equiv (s, OC)$. The conic (C) [6] may be taken to pass through A' and to be tangent to AD' at A [5]. These two new conditions may replace in [6] either the two points E, F, or the points G, G', or any two of these four points if they are real. The point A' and the tangent AD' may also replace one of the points R, R' in [7], if these points are real. Finally the elements determining the conic (C) in [6] may be combined with those determining (C) in [7], provided due regard is paid to the reality of these elements. We thus obtain a number of properties of the unicursal cubic and the solution of many construction problems, which properties and problems the reader may find it interesting to formulate.

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THE FUNCTIONAL EQUATION $f[f(x)] = g(x)$.

By G. A. Pfeiffer.*

The object of this paper is to establish certain existence theorems concerning the solution of the functional equation

$$(A) \qquad f[f(x)] = g(x),$$

where

$$g(x) \equiv a_1 x + a_2 x^2 + \cdots$$

is a given analytic function defined in the neighborhood of the origin and vanishing there and such that $|a_1| = 1$ and $a_1^n \neq 1$ for any positive integral value of n. It is easily seen that under these restrictions on $g(x)$ the equation (A) has two and only two formal solutions. It is here shown that functions $g(x)$ exist such that both of the solutions of (A) are divergent or one is divergent and the other convergent or both are convergent.

The method of proof used in this paper is the method used by the author in a paper† dealing with Schroeder's functional equation

$$\phi[f(x)] = a_1 \phi(x),$$

where the given function

$$f(x) \equiv a_1 x + a_2 x^2 + \cdots, \quad \text{where} \quad |a_1| = 1 \quad \text{and} \quad a_1^n \neq 1$$

for any positive integral value of n, is analytic about the origin. The latter equation and the equation considered in the present paper are closely connected. In particular, if the equation (A) has one divergent solution, then every formal solution of the equation

$$\phi[g(x)] = a_1 \phi(x)$$

is divergent. For, if the latter had one convergent solution, $\phi_1(x)$, then

$$f_1(x) \equiv \phi_1^{-1}[c_1 \cdot \phi_1(x)] \quad \text{and} \quad f_2(x) \equiv \phi_1^{-1}[d_1 \cdot \phi_1(x)],$$

where $c_1 = \sqrt{a_1}$, $d_1 = -\sqrt{a_1}$ and $\phi_1^{-1}(x)$ denotes the inverse function of $\phi_1(x)$, would be both convergent solutions of (A) since (symbolically)‡

$$\phi_1^{-1} c_1 \phi_1 \cdot \phi_1^{-1} c_1 \phi_1 = \phi_1^{-1} d_1 \phi_1 \cdot \phi_1^{-1} d_1 \phi_1 = \phi_1^{-1} a_1 \phi_1 = g.$$

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* Read in part before the American Mathematical Society, October 30, 1915.

† See Transactions of the Am. Math. Soc., vol. 18 (1917), p. 185.

‡ The symbol fg denotes the function $f[g(x)]$.

This contradicts the hypothesis that the equation (A) has one divergent solution. Thus, Theorem 1 or Theorem 2 of the present paper implies the first theorem of the paper mentioned above which states the existence of divergent formal solutions of the Schroeder functional equation with

$$|a_1| = 1, \quad a_1^n \neq 1, \quad n = 1, 2, 3, \cdots,$$

and a suitably given function. However, it is easily shown that the latter theorem implies neither Theorem 1 nor Theorem 2 below. To show this let

$$f_1(x) \equiv a_1 x + a_2 x^2 + \cdots, \quad |a_1| = 1, \quad a_1^n \neq 1, \quad n = 1, 2, 3, \cdots,$$

be an analytic function defined about the origin and such that the functional equation

$$\phi[f_1(x)] = a_1\phi(x)$$

has no analytic solution $\phi(x)$. Let

$$g_1(x) \equiv f_1[f_1(x)] \equiv b_1 x + b_2 x^2 + \cdots.$$

Then $g_1(x)$ is analytic about the origin and, since $b_1 = a_1^2$, $|b_1| = 1$ and $b_1^n \neq 1$ for any positive integral value of n. Further, the Schroeder functional equation

$$\phi[g_1(x)] = b_1\phi(x)$$

has no analytic solution. For, if $\phi_1(x)$ were such a solution then (symbolically)

$$\phi_1 f_1 f_1 = b_1\phi_1 \quad \text{or} \quad \phi_1 f_1 \phi_1^{-1}\phi_1 f_1 \phi_1^{-1} = a_1 a_1,$$

where again $\phi_1^{-1}(x)$ denotes the inverse of $\phi(x)$. From the last equation it follows by equating coefficients of like powers of x that

$$\phi_1 f_1 \phi_1^{-1} = a_1,$$

that is, $\phi_1(x)$ is a convergent solution of the equation

$$\phi[f_1(x)] = a_1\phi(x),$$

which is in contradiction to the definition of $f_1(x)$. Thus, although the Schroeder functional equation

$$\phi[g_1(x)] = b_1\phi(x), \quad \text{where} \quad |b_1| = 1 \quad \text{and} \quad b_1^n \neq 1$$

for any positive integral value of n, has no convergent solution, the functional equation

$$f[f(x)] = g_1(x)$$

has a convergent solution, namely $f_1(x)$.

In the case that the given function $g(x) \equiv x$, the functional equation in question reduces to the equation

$$f[f(x)] = x,$$

and it is well known that the latter equation has an infinite number of divergent solutions and an infinite number of convergent solutions. The latter equation is a special case of Babbage's functional equation

$$f^n(x) = x, \text{ where } f^n(x) \equiv f[f^{n-1}(x)].*$$

THEOREM. *There exists an analytic function* $g(x) \equiv a_1x + a_2x^2 + \cdots$, *defined in the neighborhood of the origin,* $|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, 3, \cdots$, *such that the functional equation* $f[f(x)] = g(x)$ *has no solution which is analytic about the origin, i. e., every formal solution,* $f(x) \equiv c_1x + c_2x^2 + \cdots$, *is divergent for all values of* x *different from zero.*

Proof: Let

$$g_1(x) \equiv \alpha_1 x + \alpha_2 x^2 + \cdots, \qquad |\alpha_1| = 1, \quad \alpha_1^n \neq 1, \qquad n = 1, 2, \cdots,$$

be any function of x which is analytic about $x = 0$, and let

$$\phi(x) = \gamma_1 x + \gamma_2 x^2 + \cdots$$

be any formal solution of the equation

$$\phi[\phi(x)] = g_1(x).$$

Then we have

$$\gamma_1^2 = \alpha_1, \qquad \gamma_2 = \frac{\alpha_2}{\gamma_1(1 + \gamma_1)},$$

$$\gamma_3 = \frac{\gamma_1^2(1 + \gamma_1)^2\alpha_3 - 2\gamma_1\alpha_2^2}{\gamma_1^3(1 + \gamma_1)^2(1 + \gamma_1^2)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\gamma_{n+1} = \frac{\gamma_1^i(1 + \gamma_1)^j \cdots (1 + \gamma_1^{n-1})\alpha_{n+1} + P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n)}{\gamma_1^{i+1}(1 + \gamma_1)^j \cdots (1 + \gamma_1^{n-1})(1 + \gamma_1^n)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n)$ is a polynomial in γ_1, α_i, $i = 2, 3, \cdots, n$, and where $i, j, \cdots$ are integers.

We proceed to prove the theorem by determining a set of values $[a_i]$

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* Cf. the following: A. A. Bennett, Annals of Math., vol. 17 (1915), p. 37; G. A. Pfeiffer, Amer. Jour. of Math., vol. 17 (1915), p. 421; J. F. Ritt, Annals of Math., vol. 17 (1916), p. 113. (This article is, however, not concerned with analytic solutions.); L. Leau, S. M. F. Bull., vol. 26 (1898), p. 5; E. M. Lemeray, S. M. F. Bull., vol. 26 (1898), p. 10.

As related to the present subject the reader is referred to the following papers by the writer: "On the Conformal Mapping of Curvilinear Angles. The Functional Equation $\phi[f(x)] = a_1\phi(x)$," Trans. Amer. Math. Soc., vol. 18 (1917); "On the Conformal Geometry of Analytic Arcs," Amer. Journ. of Math., vol. 17 (1915); also, to the paper by A. A. Bennett entitled "The Iteration of Functions of One Variable," Annals of Math., 2d series, vol. 17 (1915), and to the literature listed in the papers just mentioned.

for the α_i such that the a_i are the coefficients of a convergent power series, and $|a_1| = 1$ and $a_1^n \neq 1$, $n = 1, 2, \cdots$, and such that c_i, d_i $(i = 2, 3, \cdots)$, the corresponding values of γ_i when $\gamma_1 = +\sqrt{a_1}$ and $-\sqrt{a_1}$ respectively, are the coefficients of two power series each with a zero radius of convergence.

Let $F_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_{n+1})$ denote the rational integral function

$$\gamma_1^i(1+\gamma_1)^j(1+\gamma_1^2)^k\cdots(1+\gamma_1^{n-1})\alpha_{n+1} + P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n).$$

Let $a^{(1)}$ be a primitive m-th root of -1, where m is even; in particular, let

$$a^{(1)} = \cos\frac{l}{m}\pi + i\sin\frac{l}{m}\pi,$$

where l is an odd positive integer $< m$ and where m is a positive integral power of 2. Then

$$b^{(1)} = \cos\frac{l+m}{m}\pi + i\sin\frac{l+m}{m}\pi$$

is also a primitive m-th root of -1 and hence the coefficients of α_{m+1} in $F_{m+1}(a^{(1)}, \alpha_2, \cdots, \alpha_{m+1})$ and $F_{m+1}(b^{(1)}, \alpha_2, \cdots, \alpha_{m+1})$ do not vanish. Therefore there exist definite values of $\alpha_2, \cdots, \alpha_{m+1}$, say $a_2, \cdots, a_{m+1}$, such that

$$|a_i - a_i'| < \delta, \; i = 2, 3, \cdots, m+1,$$

where δ is an arbitrary positive number and the a_i', $i = 2, 3, \cdots$, are the coefficients of any convergent power series, and such that

$$F_{m+1}(t, a_2, \cdots, a_{m+1}) \neq 0$$

for both $|t - a^{(1)}| < \epsilon_1$ and $|t - b^{(1)}| < \epsilon_1$, where ϵ_1 is a positive number sufficiently small. In particular, $a_2, \cdots, a_m, a_i'$ $(i = 2, 3, \cdots)$ may all be taken equal to zero.

Let $\epsilon_1' \leqq \epsilon_1$ be a positive number such that no root of ± 1 of order $< m$ is in either of the ranges $|t - a^{(1)}| \leqq \epsilon_1'$, $|t - b^{(1)}| \leqq \epsilon_1'$. Such a number ϵ_1' obviously exists since there is only a finite number of such roots of ± 1. Since

$$\left|\frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1+t)^j(1+t^2)^k\cdots(1+t^{m-1})}\right|$$

has a lower bound $\mu_{m+1} > 0$ for both $|t - a^{(1)}| \leqq \epsilon_1'$ and $|t - b^{(1)}| \leqq \epsilon_1'$, there exists a positive number $\epsilon_1'' \leqq \epsilon_1'$, such that

$$\left|\frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1+t)^j(1+t^2)^k\cdots(1+t^{m-1})(1+t^m)}\right| > \lambda_{m+1},$$

where λ_{m+1} is an arbitrary positive number, for $0 < |t - a^{(1)}| < \epsilon_1''$ and $0 < |t - b^{(1)}| < \epsilon_1''$, $|t| = 1$, and no root of ± 1 of order $< m$ is in either of the ranges $|t - a^{(1)}| < \epsilon_1''$, $|t - b^{(1)}| < \epsilon_1''$.

Let $p > m$ be a positive integral power of 2 and let $a^{(2)}$ be a primitive p-th root of -1 such that $|a^{(2)} - a^{(1)}| < \frac{\epsilon_1''}{2}$. Such a number $a^{(2)}$ is easily determined as follows: We have

$$a^{(1)} = \cos\frac{l}{m}\pi + i\sin\frac{l}{m}\pi,$$

l = an odd positive integer $< m$ and m = a positive integral power of 2. Let p be a positive integral power of 2 and $> m$ and $\frac{2\pi}{\epsilon''}\left(\frac{\epsilon''}{2}\right.$ is here assumed to be less than unity$\Big)$ and let k be the odd positive integer which is such that

$$\frac{lp}{m} - 1 < k \leqq \frac{lp}{m} + 1.$$

Then it is easily shown that the number $\cos\frac{k}{p}\pi + i\sin\frac{k}{p}\pi$ may be taken as the number $a^{(2)}$. Also, there exists a primitive p-th root of -1, say $b^{(2)}$, such that $|b^{(2)} - b^{(1)}| < \frac{\epsilon_1''}{2}$. In particular,

$$b^{(2)} = \cos\frac{k+p}{p}\pi + i\sin\frac{k+p}{p}\pi$$

is such a number.

Now proceeding as above, there exists a positive number $\epsilon_2 < \frac{\epsilon_1''}{2}$ such that $F_{p+1}(t, a_2, \cdots, a_{p+1}) \neq 0$ for both $|t - a^{(2)}| < \epsilon_2$ and $|t - b^{(2)}| < \epsilon_2$, where the a_i, $i = 2, \cdots, m+1$ are those fixed upon above and $|a_i - a_i'| < \delta$, $i = 2, \cdots, p+1$. Again, the a_i, $i = m+2, m+3, \cdots, p$, may be all taken equal to zero. Then let $\epsilon_2' \leqq \epsilon_2$ be a positive number such that no root of ± 1 of order $< p$ is in either of the ranges

$$|t - a^{(2)}| \leqq \epsilon_2', \qquad |t - b^{(2)}| \leqq \epsilon_2'.$$

Then we have, as above,

$$\left|\frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^{e+1}(1+t)^f(1+t^2)^g\cdots(1+t^{p-1})(1+t^p)}\right| > \lambda_{p+1},$$

where λ_{p+1} is an arbitrary positive number, for

$$0 < |t - a^{(2)}| < \epsilon_2'' \quad \text{and} \quad 0 < |t - b^{(2)}| < \epsilon_2'', \quad |t| = 1,$$

where ϵ_2'' is a positive number $\leqq \epsilon_2'$, and no root of ± 1 of order $< p$ is in either of the ranges

$$|t - a^{(2)}| < \epsilon_2'', \qquad |t - b^{(2)}| < \epsilon_2''.$$

Again, let r be an even integer greater than p and let $a^{(3)}$ and $b^{(3)}$ be two primitive r-th roots of -1 such that

$$|a^{(3)} - a^{(2)}| < \frac{\epsilon_2''}{2} \qquad \text{and} \qquad |b^{(3)} - b^{(2)}| < \frac{\epsilon_2''}{2}$$

and continue as above. Thus corresponding to the terms of the infinite sequence $m, p, r, \cdots$, where $m, p, r, \cdots$ are positive integers such that $m < p < r < \cdots$, we have the inequalities

$$\left| \frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1+t)^j(1+t^2)^k \cdots (1+t^{m-1})(1+t^m)} \right| > \lambda_{m+1}$$

for both $0 < |t - a^{(1)}| < \epsilon_1''$ and $0 < |t - b^{(1)}| < \epsilon_1''$,

$$\left| \frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^{e+1}(1+t)^f(1+t^2)^g \cdots (1+t^{p-1})(1+t^p)} \right| > \lambda_{p+1}$$

for both $0 < |t - a^{(2)}| < \epsilon_2''$ and $0 < |t - b^{(2)}| < \epsilon_2''$,

$$\left| \frac{F_{r+1}(t, a_2, \cdots, a_{r+1})}{t^{u+1}(1+t)^v(1+t^2)^w \cdots (1+t^{r-1})(1+t^r)} \right| > \lambda_{r+1}$$

for both $0 < |t - a^{(3)}| < \epsilon_3''$ and $0 < |t - b^{(3)}| < \epsilon_3''$,

. . . . ,

where the λ_i are arbitrary positive numbers.

Now the range

$$|t - a^{(i)}| < \epsilon_i'', \qquad |t| = 1,$$

is contained in the range

$$|t - a^{(i-1)}| < \epsilon_{i-1}'', \qquad |t| = 1,$$

where $\epsilon_i'' < \frac{\epsilon_{i-1}''}{2}$. Consequently, there is just one number common to all of the ranges $|t - a^{(i)}| < \epsilon_i''$, $|t| = 1$. No root of ± 1 can be common to all of these ranges, since any root of ± 1 which is contained in a certain range is not contained in any of the succeeding ones. Let a be the number common to all of these ranges. Then a and $a_1 = a^2$ each has unity for its modulus, and $a^n \neq \pm 1$ and $a_1^n \neq 1$ for $n = 1, 2, 3, \cdots$. Likewise, the ranges $|t - b^{(i)}| < \epsilon_i''$, $|t| = 1$, have just one number in common, say b. Evidently

$$b^2 = a^2 = a_1 \qquad \text{and} \qquad b^n \neq \pm 1, \qquad n = 1, 2, 3, \cdots.$$

Let the positive numbers

$$\lambda_{m+1},\ \lambda_{p+1},\ \lambda_{r+1},\ \cdots$$

be taken so that the sequence

$$\sqrt[1]{\lambda_{m+1}},\ \sqrt[2]{\lambda_{p+1}},\ \sqrt[3]{\lambda_{r+1}},\ \cdots$$

is not bounded, then the sequences

$$\sqrt[1]{|c_{m+1}|},\ \sqrt[2]{|c_{p+1}|},\ \sqrt[3]{|c_{r+1}|},\ \cdots;\ \sqrt[1]{|d_{m+1}|},\ \sqrt[2]{|d_{p+1}|},\ \sqrt[3]{|d_{r+1}|},\ \cdots$$

are not bounded, and, consequently, the series

$$\sum_1^\infty c_i x^i, \qquad \sum_1^\infty d_i x^i$$

are divergent for all values of $x \neq 0$. The function $g(x)$ of the theorem is $a_1x + a_2x^2 + a_3x^3 + \cdots$. Q. E. D.

For the function $g(x)$ just determined the two formal solutions of the given functional equation are divergent. For some functions $g(x)$ there exist one convergent solution and one divergent solution. For others both solutions are convergent in the neighborhood of the origin. We proceed to prove the

THEOREM. *There exists an analytic function* $g(x) \equiv a_1x + a_2x^2 + \cdots$, *defined about the origin,* $|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, 3, \cdots$, *such that the functional equation* $f[f(x)] = g(x)$ *has one and only one solution which is analytic about the origin.*

Proof: Consider the two expressions

$$\begin{aligned} H \equiv{} & \gamma_1(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n) + \gamma_2(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n)^2 \\ & + \cdots + \gamma_n(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n)^n, \\ J \equiv{} & -\gamma_1(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n) + \gamma_2'(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n)^2 \\ & + \cdots + \gamma_n'(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n)^n, \end{aligned}$$

and then consider the set of equations obtained by equating the coefficients of like powers of x in H and J. This set of equations is as follows:

$$\begin{gathered} \gamma_1\gamma_2 + \gamma_2\gamma_1^2 = -\gamma_1\gamma_2' + \gamma_2'\gamma_1^2, \\ \gamma_1\gamma_3 + 2\gamma_1\gamma_2^2 + \gamma_3\gamma_1^3 = -\gamma_1\gamma_3' - 2\gamma_1\gamma_2'^2 - \gamma_3'\gamma_1^3, \\ \gamma_1\gamma_4 + 2\gamma_1\gamma_2\gamma_3 + \gamma_2^3 + 3\gamma_1\gamma_3^2 + \gamma_4\gamma_1^4 \\ = -\gamma_1\gamma_4' - 2\gamma_1\gamma_2'\gamma_3' + \gamma_2'^3 - 3\gamma_1\gamma_3'^2 + \gamma_4'\gamma_1^4, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{gathered}$$

$$\gamma_1\gamma_{2n} + P_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}) + \gamma_{2n}\gamma_1^{2n}$$
$$= -\gamma_1\gamma'_{2n} + P'_{2n}(\gamma_1, \gamma_2', \cdots, \gamma'_{2n-1}) + \gamma'_{2n}\gamma_1^{2n},$$

$$\gamma_1\gamma_{2n+1} + P_{2n+1}(\gamma_1, \gamma_2, \cdots, \gamma_{2n}) + \gamma_{2n+1}\gamma_1^{2n+1}$$
$$= -\gamma_1\gamma'_{2n+1} + P'_{2n+1}(\gamma_1, \gamma_2', \cdots, \gamma'_{2n}) - \gamma'_{2n+1}\gamma_1^{2n+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where P_i, P_i' are polynomials in the indicated arguments.

Solving this system of equations, we determine γ'_{2n} and γ'_{2n+1} as functions of $\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}$ and $\gamma_1, \gamma_2, \cdots, \gamma_{2n}$ respectively. We have

$$\gamma_2' = \frac{\gamma_1(1+\gamma_1)\gamma_2}{-\gamma_1(1-\gamma_1)},$$

$$\gamma_3' = \frac{\gamma_1^3(1-\gamma_1)^2(1+\gamma_1^2)\gamma_3 + \boldsymbol{P}_3(\gamma_1, \gamma_2)}{-\gamma_1^3(1-\gamma_1)^2(1+\gamma_1^2)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\gamma'_{2n} = \frac{\gamma_1^i(1-\gamma_1)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1+\gamma_1^{2n-2})(1+\gamma_1^{2n-1})\gamma_{2n} + \boldsymbol{P}_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1})}{-\gamma_1^i(1-\gamma)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1+\gamma_1^{2n-2})(1-\gamma_1^{2n-1})},$$

$$\gamma'_{2n+1} = \frac{\gamma_1^q(1-\gamma_1)^r(1+\gamma_1^2)^s(1-\gamma_1^3)^t\cdots(1-\gamma_1^{2n-1})(1+\gamma_1^{2n})\gamma_{2n+1} + \boldsymbol{P}_{2n+1}(\gamma_1, \gamma_2, \cdots, \gamma_{2n})}{-\gamma_1^q(1-\gamma_1)^r(1+\gamma_1^2)^s(1-\gamma_1^3)^t\cdots(1-\gamma_1^{2n-1})(1+\gamma_1^{2n})},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $\boldsymbol{P}_i$ are polynomials in the indicated arguments.

The preceding method of proof will now be used to show that values of $\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots$ can be found, such that $\Sigma\gamma_i x^i$ is a convergent power series for those values of γ_i, and such that the corresponding values of $\gamma_2', \gamma_4', \cdots, \gamma'_{2n}, \cdots$ as determined by the above set of equations are the coefficients of a power series with a zero radius of convergence. Thus, we consider the set of equations which determine γ'_{2n}. We write

$$F_{2n} \equiv \gamma_1^i(1-\gamma_1)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1-\gamma_1^{2n-2})(1+\gamma_1^{2n-1})\gamma_{2n}$$
$$+ \boldsymbol{P}_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}).$$

In this proof primitive roots of $+1$ of odd order are used. If $c^{(1)}$ is a primitive m-th root of $+1$, $m =$ an odd positive integer, then $(c^{(1)})^s \neq \pm 1$ for s a positive integer $< m$, and therefore the coefficient of γ_{m+1} in $F_{m+1} \neq 0$ ($m+1$, even) for $\gamma_1 = c^{(1)}$. Following the same line of argument as above, we obtain *one* set of ranges $0 < |t - c^{(1)}| < \epsilon_1''$, $0 < |t - c^{(2)}| < \epsilon_2''$, $0 < |t - c^{(3)}| < \epsilon_3''$, $\cdots$; $|t| = 1$, where $c^{(i)}$, $i = 1, 2, 3, \cdots$, are primitive roots of $+1$ of odd orders, $m, p, r, \cdots$,

respectively, and such that the order of $c^{(k)}$ is greater than the order of $c^{(j)}$ if $k > j$, which close down on one number c_1. Further, these ranges are such that $|\gamma_{2j}'| > \lambda_i$ for $0 < |t - c^{(i)}| < \epsilon_i''$, where λ_i is an arbitrary positive number and $c^{(i)}$ is a certain primitive root of $+1$ of order $2j - 1$ chosen in a manner exactly analogous to the method of choice used in the preceding proof. For definiteness we take the numbers $m, p, r, \cdots$ in this case (the notation is that used in the preceding proof) to be prime numbers > 2 and such that $m < p < r < \cdots$, and let

$$c^{(1)} = \cos\frac{l}{m}\pi + i\sin\frac{l}{m}\pi,$$

where l is an even positive integer $< m$, a prime number > 2. Then

$$c^{(2)} = \cos\frac{k}{p}\pi + i\sin\frac{k}{p}\pi,$$

where p is a prime number $> m$ and $\frac{2\pi}{\epsilon''}$ and where k is the even integer which is such that

$$\frac{lp}{m} - 1 < k \leqq \frac{lp}{m} + 1.$$

Then, again, it is readily shown that $|c^{(2)} - c^{(1)}| < \frac{\epsilon_1''}{2}$. We proceed similarly in the choice of $c^{(3)}$, a primitive r-th root of unity. The notation used here indicates that γ_{2j}' is of order index $2j$ in the sequence $[\gamma_n']$ and of order index i in the particular sub-sequence of the sequence $[\gamma_n']$ used in establishing the inequalities $|\gamma_{2j}'| > \lambda_i$. Then, as above, by properly taking the λ_i the set of values $[c_{2j}]$ of γ_{2j} is determined such that the corresponding values of γ_{2j}', c_{2j}' say, thus determined are the coefficients of a divergent power series while $\Sigma c_n x^n$ is a convergent power series, the c_n not among the chosen c_{2j} being arbitrary, except that they be the coefficients of a convergent power series; in particular, they may all be taken equal to zero.

Let

$$f_1(x) \equiv \Sigma c_n x^n,$$

then

$$g(x) \equiv f_1[f_1(x)]$$

is analytic about the origin. The other formal solution of the equation

$$f[f(x)] = g(x) \equiv f_1[f_1(x)]$$

is $\Sigma c_n' x^n$, where the c_n' are the values of γ_n' determined by the above set of equalities in γ'_n and γ_n when γ_n are put equal respectively to the cor-

responding particular values c_n just fixed upon. $\Sigma c_n' x^n$ is divergent for all values of $x \neq 0$, and

$$g(x) \equiv f_1[f_1(x)]$$

is a function as required by the theorem. Q. E. D.

To the two theorems just proved we add the following easily proved theorem as a natural completion of the above:

THEOREM. *There exists a function*

$$g(x) \equiv a_1 x + a_2 x^2 + \cdots, \qquad |a_1| = 1, \qquad a_1^n \neq 1, \qquad n = 1, 2, 3, \cdots,$$

which is analytic about the origin and which is such that the functional equation $f[f(x)] = g(x)$ *has two solutions which are analytic about the origin.*

Proof: The simplest example of a $g(x)$ which proves this theorem is the linear function $a_1 x$ (a_1 arbitrary, except that the conditions of theorem are satisfied); the two solutions are $\sqrt{a_1}x$ and $-\sqrt{a_1}x$. Non-linear examples of a $g(x)$ are easily gotten by taking the transform of the function $a_1 x$ by any analytic function,

$$b(x) \equiv b_1 x + b_2 x^2 + \cdots, \; b_1 \neq 0,$$

defined about the origin; i. e.,

$$g(x) \equiv b[a_1(b^{-1}(x))],$$

where $b^{-1}(x)$ denotes the inverse of $b(x)$, is such a function. In this case the two solutions are $b[c_1(b^{-1}(x))]$ and $b[d_1(b^{-1}(x))]$, where

$$c_1 = \sqrt{a_1}, \qquad d_1 = -\sqrt{a_1}$$

and both are analytic in the neighborhood of the origin. That either solution satisfies the given functional equation is evident. For, we have, symbolically,

$$bc_1b^{-1}bc_1b^{-1} = bd_1b^{-1}bd_1b^{-1} = ba_1b^{-1}.$$

PRINCETON UNIVERSITY.
January, 1918.

THE EXISTENCE OF THE FUNCTIONS OF THE ELLIPTIC CYLINDER.*

By Mary F. Curtis.

1. Introduction. The periodic solutions, of period 2π, of the differential equation

$$(1) \qquad \frac{d^2E(\varphi)}{d\varphi^2} + 4\left(\frac{2}{b}\cos 2\varphi + z\right)E(\varphi) = 0, \quad b \neq 0,$$

where b and z are real parameters, are known as the functions of the elliptic cylinder.† Heine was the first to attempt to establish‡ the existence of these functions. He tried to show that, for a given value of b, there exist infinitely many real values of z—the characteristic numbers—for each of which (1) has a periodic solution, not identically zero, of period 2π. The method by which he proposed to prove the existence of the characteristic numbers and thus of the functions of the elliptic cylinder is the method used in the present paper. It has already been developed by Dannacher;§ his proof is, however, involved and not evidently rigorous.

An entirely different method of determining the periodic solutions of (1) and establishing their existence was given by Bôcher|| by means of Sturm's Theorem of Oscillation. The existence of the characteristic numbers also follows as a special case of the theory of more general differential equations with periodic coefficients as given, for instance, by Bôcher¶ or Mason.**

Heine's method is, however, historically the first and it is of interest to see how it can be made simple and accurate. Furthermore, this method has furnished a feasible means of computing the actual values†† of the characteristic numbers for a given b.

* Presented to the Society, September 5, 1917.

† The equation (1) is fundamental in physical problems dealing with elliptic membranes and cylinders; see, for example, Mathieu, Journal de Liouville, 2e Serie, vol. 13 (1868), pp. 137–204; Maclaurin, Transactions of the Cambridge Philosophical Society, vol. 17′ (1898), pp. 41–109.

‡ Handbuch der Kugel funktionen I (1878/81), pp. 401–413.

§ Inaugural Dissertation, Zürich, 1906.

|| Ueber die Reihenentwicklungen der Potentialtheorie (1894), p. 181.

¶ Comptes Rendus, vol. 141 (1906), p. 928.

** Comptes Rendus, vol. 141 (1906), p. 1086.

†† Butts, American Journal of Mathematics, vol. 30 (1908), pp. 129–155.

In discussing (1) we restrict $b > 0$; the case $b < 0$ may be transformed into the case $b > 0$ by replacing b by $-b$ and φ by $\varphi + \pi/2$.

2. **The general periodic solution of period 2π.** Assume that (1) has a solution $E(\varphi)$, periodic of period 2π; then $E(\varphi)$ and its first three derivatives are continuous. Hence $E(\varphi)$ may be developed into a Fourier Series

$$E(\varphi) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty} (a_i \cos i\varphi + c_i \sin i\varphi), \tag{2}$$

which for all values of φ converges absolutely and uniformly to $E(\varphi)$* and which may be differentiated term by term twice.†

Substituting in (1) the development for $\dfrac{d^2E(\varphi)}{d\varphi^2}$, together with that for $E(\varphi)$, we have

$$\begin{aligned} &\sum_{i=1}^{\infty} i^2(a_i \cos i\varphi + c_i \sin i\varphi) \\ &\qquad = 4\left(\frac{2}{b}\cos 2\varphi + z)(\tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}(a_i \cos i\varphi + c_i \sin i\varphi)\right). \end{aligned} \tag{2'}$$

The series development for $E(\varphi)$ is absolutely convergent and it remains so on multiplication term by term by $[(2/b)\cos 2\varphi + z]$; hence in the resulting series we may remove parentheses and rearrange terms at pleasure. Thus the right hand side of (2′) may be rewritten as

$$\begin{aligned} 4\Bigg\{ &\frac{a_0}{b}\cos 2\varphi + \frac{1}{b}\sum_{i=1}^{\infty} 2(a_i \cos i\varphi \cos 2\varphi + c_i \sin i\varphi \cos 2\varphi) \\ &+ \frac{za_0}{2} + z\sum_{i=1}^{\infty}(a_i \cos i\varphi + c_i \sin i\varphi)\Bigg\} \end{aligned}$$

and again, by the use of trigonometric identities, as

$$\begin{aligned} \frac{4}{b}\Bigg\{ &a_2 + z\frac{ba_0}{2} + (a_3 + (zb+1)a_1)\cos\varphi \\ &+ \sum_{i=2}^{\infty}(a_{i-2} + a_{i+2} + zba_i)\cos i\varphi + (c_3 + (zb-1)c_1)\sin\varphi \\ &+ (c_4 + zbc_2)\sin 2\varphi + \sum_{i=3}^{\infty}(c_{i-2} + c_{i+2} + zbc_i)\sin i\varphi\Bigg\}. \end{aligned} \tag{2a}$$

We may also rearrange the terms on the left of (2′):

$$\sum_{i=1}^{\infty} i^2 a_i \cos i\varphi + \sum_{i=1}^{\infty} i^2 c_i \sin i\varphi. \tag{2b}$$

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* Bôcher, Introduction to the Theory of Fourier Series, these Annals, Series 2, vol. 7 (1906), p. 109.

† Bôcher, Loc. cit., p. 116.

The series (2*a*) converges to the same value as the series (2*b*); hence we may equate corresponding coefficients,* thereby obtaining four groups of relations, one involving the a_{2i}, one the a_{2i+1}, one the c_{2i} and one the c_{2i+1}. Since further we may write

$$E(\varphi) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_{2i} \cos 2i\varphi + \sum_{i=1}^{\infty} a_{2i-1} \cos (2i - 1)\varphi$$
$$+ \sum_{i=1}^{\infty} c_{2i} \sin 2i\varphi + \sum_{i=1}^{\infty} c_{2i-1} \sin (2i - 1)\varphi,$$

we have

THEOREM 1. *If a periodic solution, of period* 2π, *of the differential equation* (1) *exists, it is a linear combination of four particular solutions of the form*:

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi, \qquad E_2(\varphi) = \sum_{i=1}^{\infty} \beta_i \sin 2i\varphi,$$

$$E_3(\varphi) = \sum_{i=1}^{\infty} \gamma_i \cos (2i - 1)\varphi. \qquad E_4(\varphi) = \sum_{i=1}^{\infty} \delta_i \sin (2i - 1)\varphi.$$

We need therefore to consider only the existence of periodic solutions of these types and it is sufficient to confine our attention to solutions of the first type; the work for the others is quite similar.

3. **The solution** $E_1(\varphi)$.—Suppose that $E_1(\varphi)$, a real, even, periodic function of period π, is a solution of (1). It possesses then the Fourier development

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2_i\varphi. \tag{3}$$

The formal substitution of this development in (1) yields according to the first of the four groups mentioned above the recurrence relations for determining the α's:

$$\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= -\tfrac{1}{2}bz, \\
\alpha_2 &= b(1 - z)\alpha_1 - 1, \\
\alpha_3 &= b(4 - z)\alpha_2 - \alpha_1, \\
&\quad \cdots\cdots\cdots \\
\alpha_m &= b((m - 1)^2 - z)\alpha_{m-1} - \alpha_{m-2}, \\
&\quad \cdots\cdots\cdots
\end{aligned} \tag{4}$$

where, remembering that (1) is homogeneous, we have set $\alpha_0 = 1$.

* Bôcher, loc. cit., p. 151.

If $E_1(\varphi)$ is to be a solution of (1), it is necessary* that α_m, as determined by (4), approach zero as m becomes infinite. To prove this condition also sufficient we establish first two theorems which summarize completely the behavior of the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ for a given value of b.

THEOREM 2. *For any particular value of z, real or complex, the sequence $|\alpha_0|, |\alpha_1|, |\alpha_2|, \cdots$ is ultimately either continually increasing or continually decreasing.*

Given a particular value of z; choose μ so that

$$b\,|m^2 - z| > 2, \qquad m \geqq \mu. \tag{5}$$

If the terms of the sequence $|\alpha_\mu|, |\alpha_{\mu+1}|, |\alpha_{\mu+2}| \cdots$ continually decrease, our theorem is granted. If not, there exists a first α, call it α_k, $k \geqq \mu$, such that $|\alpha_{k-1}| \leqq |\alpha_k|$; then $\alpha_k \neq 0$. From (4) we have

$$\begin{aligned} |\alpha_{k+1}| &\geqq b\,|k^2 - z|\,|\alpha_k| - |\alpha_{k-1}| \\ &\geqq (b\,|k^2 - z| - 1)\,|\alpha_k| + |\alpha_k| - |\alpha_{k-1}| > |\alpha_k|. \end{aligned}$$

In like manner $|\alpha_{k+n}| > |\alpha_{k+n-1}|$, $n > 1$, and the sequence of absolute values is ultimately a continually increasing sequence. In fact we have

$$|\alpha_\mu| > |\alpha_{\mu+1}| > \cdots > |\alpha_{k-1}| \leqq |\alpha_k| < |\alpha_{k+1}| < |\alpha_{k+2}| < \cdots. \tag{6a}$$

THEOREM 3. *For any particular value of z, real or complex, the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ either approaches zero or becomes infinite.*

If the α's remain finite, $|\alpha_m| < G$ for all m, then α_m approaches zero as m becomes infinite. For, from

$$b(m^2 - z)\alpha_m = \alpha_{m+1} + \alpha_{m-1}$$

it follows that $b\,|m^2 - z|\,|\alpha_m| < 2G$. If the α's do not remain finite, they become infinite. For, the absolute value sequence does not remain finite and by theorem 2 it is from a certain term on continually increasing.

COROLLARY. *If for a particular value of z, real or complex, $\lim_{m=\infty} \alpha_m = 0$, then*

$$|\alpha_\mu| > |\alpha_{\mu+1}| > |\alpha_{\mu+2}| > \cdots \tag{6b}$$

where μ is determined as in (5).

For, if there exists a $k \geqq \mu$ such that $|\alpha_{k-1}| \leqq |\alpha_k|$, then, as in (6a), $|\alpha_k| < |\alpha_{k+1}| < \cdots$, and the sequence becomes infinite.

THEOREM 4. *A necessary and sufficient condition that $E_1(\varphi)$ is a solution of the differential equation* (1) *is that*

$$\lim_{m=\infty} \alpha_m = 0.$$

---

* Bôcher, loc. cit., p. 151.

The condition has already been proved necessary; to prove it sufficient we assume

$$\lim_{m=\infty} \alpha_m = 0.$$

Then from (4) and (6b), we have

$$\begin{aligned} |\alpha_n| &\geqq b\,|(n+1)^2 - z|\,|\alpha_{n+1}| - |\alpha_{n+2}| \\ &> (b\,|(n+1)^2 - z| - 1)\,|\alpha_{n+1}|, \qquad n \geqq \mu. \end{aligned}$$

From this inequality it can be easily inferred that there exists an l, such that

$$|\alpha_n| > bn^2\,|\alpha_{n+1}|, \qquad n > l.$$

Hence the series $\sum_{i=1}^{\infty} \alpha_n$, where the α_n are given by (4), converges absolutely. Therefore the series

$$\tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi$$

converges uniformly and represents a continuous function $E_1(\varphi)$. If we differentiate term by term, we have

$$-\,2\sum_{i=1}^{\infty} i\alpha_i \sin 2i\varphi.$$

This series converges uniformly to $\dfrac{dE_1(\varphi)}{d\varphi}$. For, in the series of the coefficients the test ratio is

$$\frac{n+1}{n}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < \frac{n+1}{bn^3}, \qquad n > l.$$

Similarly we may differentiate a second time term by term, and show that

$$-\,4\sum_{i=1}^{\infty} i^2\alpha_i \cos 2i\varphi$$

converges uniformly to $\dfrac{d^2E_1(\varphi)}{d\varphi^2}$. Hence the condition is sufficient.

4. **The existence of the ρ_i.**—Those values of z for which

$$\lim_{m=\infty} \alpha_m = 0, \qquad b = b_0,$$

are precisely the characteristic numbers. In order to establish their existence, we shall show that a sequence of suitably chosen roots, one from each of the equations $\alpha_{m+k} = 0$, $k = 1, 2, 3, \cdots$, converges to a limiting value for z and that for such a value and only for such a value

$$\lim_{m=\infty} \alpha_m = 0.$$

The α's as given by (4) are, for a fixed value of b, polynomials in z with the Sturmian properties:

1) α_0 is a constant different from zero,
2) α_m is of the mth degree in z, $\quad m = 1, 2, \cdots,$
3) α_m, α_{m+1} are relatively prime, $\quad m = 1, 2, \cdots,$
3) $\alpha_{m+1}\alpha_{m-1} < 0, \quad$ if $\quad \alpha_m = 0.$

Hence the number of real roots of $\alpha_m = 0$ in the interval $a \leqq z \leqq c$, $\alpha_m(a) \neq 0$, $\alpha_m(c) \neq 0$ is at least $| V(c) - V(a) |$, where $V(z_0)$ denotes the number of variations in sign of the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$ for $z = z_0$. Since

$$V(-\infty) = 0, \qquad V(\infty) = m,$$

$\alpha_m = 0$ has at least m distinct, real roots and therefore exactly m. For, α_m is of the mth degree; accordingly $V(z)$ must increase by unity each time z increasing passes through a root of $\alpha_m = 0$ and we have the following lemma and theorem:

Sturmian Lemma. The number of real roots of $\alpha_m = 0$ in the interval

$$a \leqq z \leqq c, \qquad \alpha_m(a) \neq 0, \qquad \alpha_m(c) \neq 0,$$

is precisely $V(c) - V(a)$.

THEOREM 5. *The m roots of $\alpha_m = 0$ are all real and distinct.*

The m roots of $\alpha_m = 0$, $m > 1$, are separated by the $m - 1$ roots of $\alpha_{m-1} = 0$, that is,

THEOREM 6. *Between each two successive roots of $\alpha_m = 0$, $m > 1$, $\alpha_{m-1} = 0$ has one and only one root.*

For, suppose that in the interval $z_1 < z < z_2$, where z_1, z_2 are two successive roots of

$$\alpha_m = 0, \qquad m > 1,$$

$\alpha_{m-1} = 0$ has no root; then in the interval $z_1 - \epsilon < z < z_1 + \epsilon$, for ϵ sufficiently small and positive, α_{m-1} maintains its sign while α_m changes sign twice. Thus

$$V(z_2 + \epsilon) - V(z_1 - \epsilon) = 0$$

and a contradiction to the lemma first stated is established. Therefore $\alpha_{m-1} = 0$ has at least one root between z_1 and z_2 and certainly not more than one.*

* From Dannacher's work it seems improbable that he knew of Theorem 6. Had he made use of it, his proof of the existence of the ρ_i would have been much simplified.

If $b \geqq 1$, $V(-2) = 0$; if $0 < b < 1$, $V\left(\frac{-2}{b}\right) = 0$. Hence every root of $\alpha_m = 0$, $m = 1, 2, \cdots$, is greater than -2, $-2/b$, according as $b \geqq 1$, $0 < b < 1$. Number the roots of $\alpha_m = 0$ in natural order beginning with the smallest and denote the ith root, $i \leqq m$, of $\alpha_m = 0$ by $r_{m,\, i}$. Since, by Theorem 6, the $m + 1$ roots of $\alpha_{m+1} = 0$ are separated by the m roots of $\alpha_m = 0$, $r_{m+1,\, i} < r_{m,\, i}$; similarly $r_{m+n+1,\, i} < r_{m+n,\, i}$, $n > 1$. Hence

$$r_{m,\, i} > r_{m+1,\, i} > r_{m+2,\, i} > \cdots > -2, \qquad -2/b$$

and the ith roots of $\alpha_m = 0$, $\alpha_{m+1} = 0$, $\cdots$ form a decreasing sequence approaching a limiting value ρ_i.

THEOREM 7. *The ith root, $i \leqq m$, of $\alpha_{m+k} = 0$ decreases as k increases and approaches a limiting value ρ_i as k becomes infinite.**

5. **Heine's "fundamental error."**—It is in showing that the ρ_i are the characteristic numbers, that is, are the values of z for which $\lim\limits_{m=\infty} \alpha_m = 0$, that Heine makes his "fundamental error."† He assumes forthwith that the ρ_i are identical with the roots of the limiting function of the α-sequence. The conditions under which this is true were given later by Hurwitz‡ in the theorem: If the sequence of functions $f_1(z)$, $f_2(z)$, $\cdots$ converges uniformly in a finite domain C of the complex variable z to the function $f(z)$, not identically zero, and $f_i(z)$, $i = 1, 2, \cdots$, behaves like a rational function in the interior of C, then the roots of $f(z)$ in the interior of C are identical with the cluster points of the roots of $f_i(z)$ in the interior of C. We can not apply this theorem directly to the α-sequence, since, as a matter of fact, the α-sequence converges only for isolated values of z. However, we shall show that, if we set

$$(7) \qquad \alpha_1 = b\beta_1, \qquad \alpha_m = b^m(m-1)!^2\beta_m, \qquad m > 1,$$

the sequence of β's thus obtained converges and converges uniformly in any finite domain C to a function β not identically zero. The ρ_i which are obviously the cluster points of the roots of the β_i as well as of the roots of the α_i will therefore, by Hurwitz's theorem, be identical with the roots of β. If then we show that $\lim\limits_{m=\infty} \beta_m = 0$ when and only when $\lim\limits_{m=\infty} \alpha_m = 0$, it will follow that the roots of $\beta = 0$, the ρ_i, are the characteristic numbers.

* It is a simple matter to determine, as, for instance, Dannacher has done, the distribution of the ρ_i and to show that for m sufficintly large,

$$(m-1)^2 < \rho_m < (m-1)^2 + \frac{1}{b\{b(2m-3)-1\}} < m^2.$$

† Dannacher, loc. cit., p. 1.

‡ Mathematische Annalen, vol. 33 (1889), pp. 246–249.

6. **The β-sequence.**—Substituting (7) in (4) we have, as the recurrence relations for determining the β's:

$$
\begin{aligned}
&\beta_1 = -\frac{z}{2},\\
(8)\qquad &\beta_2 = (1 - z)\beta_1 - \frac{1}{b^2},\\
&\beta_{m+1} = \left(1 - \frac{z}{m^2}\right)\beta_m - \frac{\beta_{m-1}}{b^2(m-1)^2m^2}, \qquad m > 1.
\end{aligned}
$$

We first prove:

THEOREM 8. *A necessary and sufficient condition that*

$$\lim_{m=\infty} \beta_m = 0$$

is that

$$\lim_{m=\infty} \alpha_m = 0.$$

Given

$$\lim_{m=\infty} \alpha_m = 0;$$

then from (7)

$$\lim_{m=\infty} b^m(m-1)!^2\beta_m = 0$$

and

$$\lim_{m=\infty} \beta_m = 0,$$

since, even when $b < 1$, $b^m(m-1)!^2$ becomes infinite with m.

Given

$$\lim_{m=\infty} \beta_m = 0.$$

Suppose

$$\lim_{m=\infty} \alpha_m \neq 0;$$

then the α-sequence becomes infinite and, as we saw in the proof of Theorem 2,

$$|\alpha_{n+1}| \geqq (b\,|n^2 - z| - 1)\,|\alpha_n|, \qquad n \geqq k,$$

where the equality sign holds at most for $n = k$, k determined as in (6a). Restrict k further, if necessary, so that

$$(9)\qquad k^2 > |z| + \frac{1}{b};$$

then

$$|\alpha_{n+1}| \geqq (bn^2 - b\,|z| - 1)\,|\alpha_n|, \qquad n \geqq k.$$

By (7) this relation yields

$$|\beta_{k+1}| \geqq \left(1 - \frac{|z| + 1/b}{k^2}\right) |\beta_k|,$$

$$|\beta_{k+2}| > \left(1 - \frac{|z| + 1/b}{(k+1)^2}\right) |\beta_{k+1}|,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$|\beta_{k+s}| > \left(1 - \frac{|z| + 1/b}{(k+s-1)^2}\right) |\beta_{k+s-1}|,$$

whence, on multiplication,

$$|\beta_{k+s}| > |\beta_k| \prod_{r=k}^{k+s-1} \left(1 - \frac{|z| + 1/b}{r^2}\right) > |\beta_k| \prod_{r=k}^{\infty} \left(1 - \frac{|z| + 1/b}{r^2}\right).$$

For, by (9)

$$0 < \frac{|z| + 1/b}{r^2} < 1, \qquad r \geqq k.$$

Since $\sum_{r=k}^{\infty} \frac{|z| + 1/b}{r^2}$ converges absolutely, the infinite product converges* to a value $c > 0$. Hence $|\beta_{k+s}| > c\,|\beta_k|$, $s = 1, 2, \cdots$. But $|\beta_k| \neq 0$, for $|\alpha_k| \neq 0$; therefore $\lim_{m=\infty} \beta_m \neq 0$ and a contradiction is established.

As a preliminary to the proof that the β-sequence converges uniformly, we prove

THEOREM 9. *In every finite domain C the β-sequence remains uniformly finite.*

Without loss of generality we may restrict C to be a circular region, $|z| < R$, with center in the origin. Given a constant $R > 0$, we have to show that a constant M exists, such that $|\beta_n| < M$ for all $|z| < R$ and for all n, or what is sufficient in this case, that a constant M exists, such that $|\beta_n| < M$ for all $|z| < R$ and for all $n \geqq l$, where l is an integer independent of z. We may take $l = \mu$, where μ is determined so that (5) holds for all z in C. For this purpose it suffices to take

$$\mu^2 > \frac{2}{b} + R. \tag{10}$$

1) For every value of z in C for which $\lim_{m=\infty} \alpha_m = 0$, (6$b$) holds and by (7)

$$|\beta_\mu| > b\mu^2 |\beta_{\mu+1}| > b^2\mu^2(\mu+1)^2 |\beta_{\mu+2}| > \cdots.$$

Since by (10) $b\mu^2 > 2$, $|\beta_\mu| > |\beta_{\mu+1}> \cdots$ and

$$|\beta_m| < |\beta_\mu| \text{ for all } m > \mu. \tag{11}$$

* Osgood, Funktiontheorie 1 (1907), p. 459; 2d ed. (1912), p. 528.

2) We want a corresponding relation for every value of z in C for which the α-sequence becomes infinite. For such a va'ue of z (6b):

$$|\alpha_\mu| > |\alpha_{\mu+1}| > \cdots > |\alpha_{k-1}| \leqq |\alpha_k| < |\alpha_{k-1}| < \cdots$$

holds, where the μ, but not the k, is independent of z. Since

$$|\alpha_\mu| > |\alpha_{\mu+1}| > \cdots > |\alpha_{k-1}|,$$

it follows as in (11) that

$$(a) \qquad |\beta_m| < |\beta_\mu|, \text{ for } m = \mu + 1, \cdots, k - 1.$$

To compare $|\beta_k|$ with $|\beta_\mu|$, we first restrict μ, if actually necessary, so that $\mu^2 > 4$; then, on the substitution of (7) in

$$|\alpha_k| \leqq (b(k-1)^2 + b|z|)|\alpha_{k-1}| + |\alpha_{k-2}|,$$

we obtain

$$|\beta_k| \leqq \left(1 + \frac{|z|}{(k-1)^2}\right)|\beta_{k-1}| + \frac{1}{b^2(k-1)^2(k-2)^2}|\beta_{k-2}|$$

$$< \left(1 + R + \frac{1}{b^2}\right)|\beta_\mu|.$$

Hence

$$(b) \qquad |\beta_k| < K|\beta_\mu|, \qquad \text{where} \qquad K = 1 + R + \frac{1}{b^2}.$$

To compare $|\beta_m|$, $m > k$, with $|\beta_\mu|$, we compare $|\beta_m|$ first with $|\beta_k|$. From

$$|\alpha_{k-1}| \leqq |\alpha_k| < \cdots,$$

we have by (4)

$$|\alpha_{n+1}| \leqq (b(n^2 + R) + 1)|\alpha_n|,$$

where the equality sign holds at most for $n = k$. By (7) this relation yields

$$|\beta_{k+1}| \leqq \left(1 + \frac{R + 1/b}{k^2}\right)|\beta_k|,$$

$$|\beta_{k+2}| < \left(1 + \frac{R + 1/b}{(k+1)^2}\right)|\beta_{k+1}|,$$

$$\cdots\cdots\cdots\cdots$$

$$|\beta_{k+s}| < \left(1 + \frac{R + 1/b}{(k+s-1)^2}\right)|\beta_{k+s-1}|,$$

and, by multiplication,

$$|\beta_{k+s}| < |\beta_k| \sum_{r=k}^{k+s-1}\left(1 + \frac{R + 1/b}{r^2}\right) < |\beta_k| \prod_{r=1}^{\infty}\left(1 + \frac{R + 1/b}{r^2}\right).$$

The infinite product converges to a value $L > 1$; hence

$$(c) \qquad |\beta_m| < L|\beta_k| < G|\beta_\mu|, \quad m \geqq k+1, \quad G = KL > 1.$$

The three inequalities (*a*), (*b*), (*c*) may evidently be combined in the one inequality

$$(12) \qquad |\beta_m| < G|\beta_\mu|, \text{ for all } m > \mu.$$

Since $G > 1$, (12) includes (11) and holds for all $|z| < R$. In C $|\beta_\mu|$ has an upper limit H and hence there exists a constant $M = GH$ such that

$$(13) \qquad |\beta_m| < M \text{ for all } m \geqq \mu \text{ and all } |z| < R.$$

With this inequality established it is a simple matter to prove

THEOREM 10. *In every finite domain C the β sequence converges uniformly to a function β, not identically zero.*

We again take C as the circular region $|z| < R$. The β-sequence converges uniformly in C if the series

$$\beta_1 + (\beta_2 - \beta_1) + (\beta_3 - \beta_2) + \cdots$$

converges uniformly in C. The general term of this series is by (8)

$$\beta_{n-1} - \beta_n = \frac{-z}{n^2}\beta_m - \frac{\beta_{n-1}}{b^2(n-1)^2n^2};$$

hence by (13)

$$|\beta_{n+1} - \beta_n| < \frac{1}{n^2}\left\{R|\beta_n| + \frac{|\beta_{n-1}|}{b^2|n-1|^2}\right\} < \frac{M(R + 1/b^2)}{n^2}, \quad n \geqq \mu.$$

The series

$$M\left(R + \frac{1}{b^2}\right)\sum_{n=\mu}^{\infty}\frac{1}{n^2}$$

forms a majorante for the β-series; therefore the β-sequence converges uniformly in C to a function β.

If

$$\beta \equiv 0, \qquad \lim_{n=\infty} \alpha_m \equiv 0;$$

but it is easily shown that for $z = n^2$, n an integer at least as great as μ, the α-sequence becomes infinite; thereby a contradiction is established.

7. **Conclusion.** According to § 5 we may now conclude the existence of the characteristic numbers and their identity with the ρ_i and the roots of $\beta = 0$.

THEOREM 11. *For a given value of b there exist infinitely many characteristic numbers, the cluster points of the roots of the equation $\alpha_m = 0$;*

for each of these infinitely many real values of z, and only for these values, (1) *possesses a periodic solution, not identically zero, of period π, of the form*

$$E_1(\varphi) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty} \alpha_i \cos 2i\varphi.$$

A similar theorem holds for periodic solutions of (1) of the forms $E_2(\varphi), E_3(\varphi), E_4(\varphi)$.

WESTERN RESERVE UNIVERSITY,
November 6, 1917.

THE GAMMA FUNCTION IN THE INTEGRAL CALCULUS.

By T. H. Gronwall.

Table of Contents.

Introduction. The object of this paper is to give an exposition, as elementary as possible, of some of those aspects of the theory of the Gamma function which are not dealt with in Jensen's "An elementary exposition of the theory of the Gamma function."[1] Chapter I presents briefly

[1] Authorized translation, with additional notes, by T. H. Gronwall. These *Annals*, ser. 2, vol. 17 (1916), pp. 124–166. References to paragraphs and footnotes in this paper will be given thus: J § 2 and J².

those definitions and theorems on definite integrals which are indispensable in the following. In chapter II, the classical applications of the integral calculus to the Gamma function are set forth in a form which adheres quite closely to the point of view of Jensen's paper, while in chapter III the same body of theorems is derived from the definition of $\Gamma(s)$ as a definite integral.*

While the paper contains little that is new in subject matter, a considerable number of proofs have been remodeled, or replaced by new ones.

CHAPTER I.

Definite Integrals.

1. **Uniform continuity.** Let us consider a function $f(x_1, x_2, \cdots, x_n)$ defined for all values of $x_1, x_2, \cdots, x_n$ belonging to a pointset P which is bounded (i. e., there exists an A such that $|x_1| < A$, $|x_2| < A$, $\cdots$, $|x_n| < A$ for all points of P). The function is *continuous* on the pointset P when, for any point $x_1, x_2, \cdots, x_n$ of P and any ϵ (arbitrarily small), there exists a $\delta = \delta(\epsilon, x_1, x_2, \cdots, x_n)$ such that for every point $x_1', x_2', \cdots, x_n'$ of P where

$$(1) \qquad |x_1' - x_1| < \delta, \quad |x_2' - x_2| < \delta, \quad \cdots, \quad |x_n' - x_n| < \delta$$

we have

$$(2) \qquad |f(x_1', x_2', \cdots, x_n') - f(x_1, x_2, \cdots, x_n)| < \epsilon.$$

The function is *uniformly continuous on the pointset* P when, for any ϵ, a $\delta = \delta(\epsilon)$ may be chosen *independently* of $x_1, x_2, \cdots, x_n$ such that the inequality (2) holds for any two points $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ of P satisfying (1). The following theorem is of fundamental importance for the definition of an integral:

THEOREM I. *When $f(x_1, x_2, \cdots, x_n)$ is continuous on a* CLOSED[2] *pointset P, this function is also uniformly continuous on P.*

Suppose the theorem to be false; then there exist an $\epsilon > 0$, an infinite sequence of pairs of points on P: $x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu}$ and $x_{1\nu}', x_{2\nu}', \cdots, x_{n\nu}'$ $(\nu = 1, 2, 3, \cdots)$ such that

$$(3) \qquad |x_{1\nu}' - x_{1\nu}| < \delta_\nu, \quad \cdots, \quad |x_{n\nu}' - x_{n\nu}| < \delta_\nu, \quad \delta_\nu \to 0 \text{ as } \nu \to \infty,$$

and for which

$$(4) \qquad |f(x_{1\nu}', x_{2\nu}', \cdots, x_{n\nu}') - f(x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu})| \geqq \epsilon.$$

Let a limiting point of the sequence $x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu}$ be $x_{10}, x_{20}, \cdots, x_{n0}$; this point belongs to P, since P is closed, and $f(x_1, x_2, \cdots, x_n)$ being

* See the note at the end of this paper.

[2] I. e., a pointset containing all its limiting points.

continuous on P, there exists a δ_0 such that for $|x_1 - x_{10}| < \delta_0$, $|x_2 - x_{20}| < \delta_0, \cdots, |x_n - x_{n0}| < \delta_0$ we have $|f(x_1, x_2, \cdots, x_n) - f(x_{10}, x_{20}, \cdots, x_{n0})| < \epsilon/2$. We may now choose ν so large that $\delta_\nu < \delta_0/2$ and $|x_{1_\nu} - x_{10}| < \delta_0/2, \cdots, |x_{n_\nu} - x_{n0}| < \delta_0/2$; hence, by (3), $|x_{1_\nu}' - x_{10}| \leqq |x_{1_\nu} - x_{10}| + |x_{1_\nu}' - x_{1_\nu}| < \delta_0/2 + \delta_\nu < \delta_0, \cdots, |x_{n_\nu}' - x_{n0}| < \delta_0$. We have therefore $|f(x_{1_\nu}, x_{2_\nu}, \cdots, x_{n_\nu}) - f(x_{10}, x_{20}, \cdots, x_{n0})| < \epsilon/2$, $|f(x_{1_\nu}', x_{2_\nu}', \cdots, x_{n_\nu}') - f(x_{10}, x_{20}, \cdots, x_{n0})| < \epsilon/2$, whence by addition $|f(x_{1_\nu}', x_{2_\nu}', \cdots, x_{n_\nu}') - f(x_{1_\nu}, x_{2_\nu}, \cdots, x_{n_\nu})| < \epsilon$, which contradicts (4), and this contradiction proves our theorem.

2. **Upper and lower integrals of a function.** We define the interval (a, b), where $a < b$, as the set of all points x such that $a \leqq x \leqq b$. This pointset is evidently closed; a and b are called the end points, and the points x such that $a < x < b$ the interior points of the interval. The *length* of the interval is $b - a$.

A set of values of a real variable ξ is said to be bounded *below*, when there exists a finite A such that $\xi > A$ for every ξ in the set, and bounded *above*, when $\xi < B$ for every ξ. It is a well-known theorem that every set of ξ's which is bounded below has a *lower bound* m such that $\xi \geqq m$ for every ξ but that for any positive ϵ there exists at least one ξ in the set such that $\xi < m + \epsilon$; similarly every set which is bounded above has an *upper bound* M such that $\xi \leqq M$ for every ξ but $\xi > M - \epsilon$ for at least one ξ in the set. It is evident that the lower and upper bounds of $-\xi$ are $-M$ and $-m$ respectively.

In particular, let ξ be the set of values assumed by a function $f(x)$ when x varies in the interval (a, b); when this set of ξ's is bounded both above and below, $f(x)$ is said to be *bounded* in the interval (a, b), and m and M are called the lower and upper bounds of $f(x)$ in this interval.

Now consider a function $f(x)$, bounded in the interval (a, b), and let us divide this interval into a number of subintervals $(x_{\nu-1}, x_\nu)$ (of length $x_\nu - x_{\nu-1}$) by means of the points of division $x_0, x_1, \cdots, x_n$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Denote by M_ν the upper, and by m_ν the lower bound of $f(x)$ in the interval $(x_{\nu-1}, x_\nu)$, and write

$$S = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) = \Sigma M_\nu(x_\nu - x_{\nu-1}),$$

$$s = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) = \Sigma m_\nu(x_\nu - x_{\nu-1}).$$

Evidently $M \geqq M_\nu \geqq m_\nu \geqq m$, and hence

$$M(b - a) \geqq S \geqq s \geqq m(b - a). \tag{5}$$

If we consider all possible values of S and s corresponding to all possible modes of subdivision of (a, b), S and s are bounded above and below by (5); hence there exist an upper bound j of s and a lower bound J of S. We shall now prove

THEOREM II. *When the number of subintervals is increased indefinitely in such a manner that the length of each tends toward zero, the sums S and s tend toward the limits J and j respectively.*

Let us change the sum s into s' by introducing a new division point x' between $x_{\nu-1}$ and x_ν. Denoting by m_ν' and m_ν'' the lower bounds of $f(x)$ in $(x_{\nu-1}, x')$ and (x', x_ν), we obviously have

$$\begin{aligned} s' - s &= m_\nu'(x' - x_{\nu-1}) + m_\nu''(x_\nu - x') - m_\nu(x_\nu - x_{\nu-1}) \\ &= (m_\nu' - m_\nu)(x' - x_{\nu-1}) + (m_\nu'' - m_\nu)(x_\nu - x'), \end{aligned}$$

and since $0 \leqq m_\nu' - m_\nu \leqq M - m$, $0 \leqq m_\nu'' - m_\nu \leqq M - m$, we find $0 \leqq s' - s \leqq (M - m)(x_\nu - x_{\nu-1}) < (M - m)\delta$ if $x_\nu - x_{\nu-1} < \delta$. More generally, if s' is obtained from s by the introduction of k new division points, we find by introducing them one at a time and successive application of the above inequality that

$$0 \leqq s' - s < k(M - m)\delta, \tag{6}$$

if every subinterval corresponding to s is less than δ. Since $-J$ and $-S$ stand in the same relation to $-f(x)$ as j and s to $f(x)$, it is sufficient to show that for any $\epsilon > 0$ there exists a δ such that $j \geqq s > j - \epsilon$ for any s in which the length of each subinterval is less than δ. Since j is the upper bound of all s, it is clear that $j \geqq s$ and that there exists an s_0 such that $s_0 > j - \epsilon/2$. Let the number of subdivisions corresponding to s_0 be k, and consider any s in which the length of each subdivision is less than

$$\delta = \frac{\epsilon}{2k(M - m)}.$$

Finally let s' correspond to the subdivision having as division points all those occurring in s and in s_0; then, by (6),

$$s > s' - k(M - m)\delta = s' - \epsilon/2,$$

and applying (6) to s_0 and s', we find $s' \geqq s_0$, so that

$$s > s_0 - \epsilon/2 > (j - \epsilon/2) - \epsilon/2 = j - \epsilon,$$

which proves our theorem.

The numbers J and j are called the upper and lower integrals of $f(x)$ between the limits a and b, and are denoted by

$$J = \overline{\int_a^b} f(x)dx, \qquad j = \underline{\int_a^b} f(x)dx.$$

Since $S \geqq s$, it follows from theorem II that $J \geqq j$ or

$$\overline{\int_a^b} f(x)dx \geqq \underline{\int_a^b} f(x)dx. \tag{7}$$

Let $a < c < b$ and consider the sums S and s for which c is a division point, it follows immediately from theorem II that

$$
(8)\qquad
\begin{aligned}
\overline{\int_a^b} f(x)dx &= \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx,\\
\underline{\int_a^b} f(x)dx &= \underline{\int_a^c} f(x)dx + \underline{\int_c^b} f(x)dx,
\end{aligned}
$$

and from (5) and (7) it is seen that

$$
(9)\qquad m(b-a) \leqq \underline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x)dx \leqq M(b-a).
$$

Furthermore, when k is a constant, the sums S and s for $kf(x)$ equal k times those for $f(x)$, whence

$$
(10)\qquad
\begin{aligned}
\overline{\int_a^b} kf(x)dx &= k\overline{\int_a^b} f(x)dx, & \underline{\int_a^b} kf(x)dx &= k\underline{\int_a^b} f(x)dx \text{ for } k \geqq 0,\\
\overline{\int_a^b} kf(x)dx &= k\underline{\int_a^b} f(x)dx, & \underline{\int_a^b} kf(x)dx &= k\overline{\int_a^b} f(x)dx \text{ for } k \leqq 0.
\end{aligned}
$$

Finally, let $f_1(x), f_2(x), \cdots, f_n(x)$ all be bounded in (a, b), and let M_ν and m_ν, $M_{1\nu}$ and $m_{1\nu}$, $\cdots$, $M_{n\nu}$ and $m_{n\nu}$ be the upper and lower bounds of $(f_1(x) + f_2(x) + \cdots + f_n(x))$, $f_1(x)$, $\cdots$, $f_n(x)$ in $(x_{\nu-1}, x_\nu)$. Evidently $M_\nu \leqq M_{1\nu} + M_{2\nu} + \cdots + M_{n\nu}$ and $m_\nu \geqq m_{1\nu} + m_{2\nu} + \cdots + m_{n\nu}$, whence

$$
\Sigma M_\nu(x_\nu - x_{\nu-1}) \leqq \Sigma M_{1\nu}(x_\nu - x_{\nu-1}) + \cdots + \Sigma M_{n\nu}(x_\nu - x_{\nu-1})
$$

and

$$
\Sigma m_\nu(x_\nu - x_{\nu-1}) \geqq \Sigma m_{1\nu}(x_\nu - x_{\nu-1}) + \cdots + \Sigma m_{n\nu}(x_\nu - x_{\nu-1}).
$$

Passing to the limit, we obtain

$$
(11)\qquad
\begin{aligned}
&\overline{\int_a^b} (f_1(x) + f_2(x) + \cdots + f_n(x))dx\\
&\qquad\qquad \leqq \overline{\int_a^b} f_1(x)dx + \overline{\int_a^b} f_2(x)dx + \cdots + \overline{\int_a^b} f_n(x)dx,\\
&\underline{\int_a^b} (f_1(x) + f_2(x) + \cdots + f_n(x))dx\\
&\qquad\qquad \geqq \underline{\int_a^b} f_1(x)dx + \underline{\int_a^b} f_2(x)dx + \cdots + \underline{\int_a^b} f_n(x)dx.
\end{aligned}
$$

3. The definite integral. When the function $f(x)$ is bounded in the interval (a, b) and the upper and lower integrals of $f(x)$ between the limits a and b are equal, the function is said to be *integrable* in (a, b), and the common value of the upper and lower integrals is called the integral

of $f(x)$ between the limits a and b and is denoted by

$$\int_a^b f(x)dx.$$

The necessary and sufficient condition for integrability is obviously that the difference (which is $\geqq 0$ by (5))

$$S - s = \Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$$

shall tend toward zero for any one particular mode of subdivision of (a, b) when the number of subdivisions is increased indefinitely in such a manner that the length of each approaches zero.

We shall now prove

THEOREM III. *A bounded function $f(x)$ is integrable in the interval (a, b) when it is possible to find a finite number of subintervals, the total length of which is as small as we please, and such that every point of discontinuity of $f(x)$ is interior to one of these subintervals.*

We may note that the hypothesis of this theorem is fulfilled when $f(x)$ has a finite number of points of discontinuity in (a, b).

To prove our theorem, it suffices, by the remark above, to exhibit a subdivision such that the corresponding sums S and s differ by less than any assigned ϵ. By hypothesis, we can find a finite number of intervals Δ', of total length less than $\dfrac{\epsilon}{2(M - m)}$, such that every point of discontinuity of $f(x)$ is interior to one of them. Denote by Δ'' the finite number of intervals obtained by removing from (a, b) the *interior* points of all intervals Δ', and consider a subdivision of (a, b) in which the end points of all intervals Δ' are among the division points. Then we may write

$$S - s = \Sigma'(M_\nu - m_\nu)(x_\nu - x_{\nu-1}) + \Sigma''(M_\nu - m_\nu)(x_\nu - x_{\nu-1}),$$

where Σ' and Σ'' refer to the intervals belonging to Δ' and Δ'' respectively. For the former, we have, since $M_\nu - m_\nu \leqq M - m$,

$$\Sigma'(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$$
$$\leqq (M - m)\Sigma'(x_\nu - x_{\nu-1}) < (M - m) \cdot \frac{\epsilon}{2(M - m)} = \frac{\epsilon}{2}.$$

Regarding Σ'', we observe that Δ'' is a closed pointset on which $f(x)$ is continuous, and by theorem I we may assign a δ so small that

$$|f(x') - f(x'')| < \frac{\epsilon}{2(b - a)}$$

for any x', x'' both belonging to Δ'' and such that $|x' - x''| < \delta$. Now

make every subdivision in Δ'' so small that $x_\nu - x_{\nu-1} < \delta$, and let x', x'' be values of x in the interval $(x_{\nu-1}, x_\nu)$ for which the continuous function $f(x)$ takes its maximum and minimum values M_ν and m_ν respectively. Then

$$M_\nu - m_\nu < \frac{\epsilon}{2(b-a)}$$

for every subinterval belonging to Δ'', and the total length of these subintervals not exceeding $b - a$, we have

$$\Sigma'' < \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2}.$$

Consequently

$$0 \leqq S - s < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves our proposition.

THEOREM IV. *When $f(x)$ is bounded and integrable in (a, b), the same is true of* $|f(x)|$.

It is evident that $|f(x)|$ is bounded when $f(x)$ is bounded. Let M_ν and m_ν, M_ν' and m_ν' be the upper and lower bounds of $f(x)$ and $|f(x)|$ in $(x_{\nu-1}, x_\nu)$. When $M_\nu \geqq m_\nu \geqq 0$, we have $M_\nu' = M_\nu$, $m_\nu' = m_\nu$; when $0 \geqq M_\nu \geqq m_\nu$, then $M_\nu' = -m_\nu$, $m_\nu' = -M_\nu$, and when $M_\nu \geqq 0 \geqq m_\nu$, M_ν' equals the greater of M_ν and $-m_\nu$, while $m_\nu' = 0$, so that in all three cases $0 \leqq M_\nu' - m_\nu' \leqq M_\nu - m_\nu$. Since $\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$ tends toward zero, the same is true of $\Sigma(M_\nu' - m_\nu')(x_\nu - x_{\nu-1})$, so that the upper and lower integrals of $|f(x)|$ are equal.

From (9) it follows that when $f(x)$ is integrable in (a, b)

$$m(b-a) \leqq \int_a^b f(x)dx \leqq M(b-a). \tag{12}$$

More generally, when three bounded and integrable functions $f(x)$, $m(x)$ and $M(x)$ satisfy the inequalities $m(x) \leqq f(x) \leqq M(x)$ in (a, b), we have

$$\int_a^b m(x)dx \leqq \int_a^b f(x)dx \leqq \int_a^b M(x)dx. \tag{13}$$

In fact, since $m(x) \leqq f(x) \leqq M(x)$ in $(x_{\nu-1}, x_\nu)$, the lower bound of $m(x)$ in this interval does not exceed that of $f(x)$, and the upper bound of $f(x)$ does not exceed that of $M(x)$; hence the sum s relative to $m(x)$ does not exceed that relative to $f(x)$, and the S relative to $f(x)$ does not exceed that relative to $M(x)$. Passing to the limit, we obtain (13).

In particular, theorem IV allows us to take $m(x) = -|f(x)|$,

$M(x) = |f(x)|$, and (13) gives

$$(14) \qquad \left| \int_a^b f(x)dx \right| \leqq \int_a^b |f(x)| \, dx.$$

When $a < c < b$, (8) becomes for an integrable $f(x)$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

and the repeated application of this formula gives

$$(15) \qquad \int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)dx + \int_{x_{n-1}}^b f(x)dx$$

for $a < x_1 < x_2 < \cdots < x_{n-1} < b$.

It is sometimes convenient to consider the case where the upper limit in an integral is smaller than the lower one; in this case we introduce the definition

$$(16) \qquad \int_b^a f(x)dx = -\int_a^b f(x)dx, \qquad a < b.$$

From (16) it follows that (15) is valid when $a, x_1, x_2, \cdots, x_{n-1}, b$ are in any order of magnitude, provided $f(x)$ is integrable in an interval containing them all. When $b < a$, the inequalities (12), (13), (14) must however be reversed.

The inequalities (10) and (11) give when $f(x), f_1(x), \cdots, f_n(x)$ are integrable

$$(17) \qquad \int_a^b kf(x)dx = k\int_a^b f(x)dx,$$

$$(18) \qquad \int_a^b (f_1(x) + f_2(x) + \cdots + f_n(x))dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx.$$

Theorem V. *When $f_1(x)$ and $f_2(x)$ are bounded and integrable in (a, b), the same is true of their product $f(x) = f_1(x)f_2(x)$.*

Let M and m, M' and m', M'' and m'' be the upper and lower bounds of $f(x), f_1(x)$ and $f_2(x)$ in (a, b); furthermore, let M_ν and m_ν, M_ν' and m_ν', M_ν'' and m_ν'' be the corresponding bounds in $(x_{\nu-1}, x_\nu)$. Evidently $|f(x)|$ does not exceed the greatest of the quantities $M'M''$, $m'm''$, $-M'm''$, $-m'M''$, so that $f(x)$ is bounded in (a, b). To prove the integrability, let us first suppose that $f_1(x)$ and $f_2(x)$ (and consequently $f(x)$) are never

negative in (a, b). Then obviously $M_\nu \leqq M_\nu' M_\nu''$, $m_\nu \geqq m_\nu' m_\nu''$, and

$$M_\nu - m_\nu \leqq M_\nu' M_\nu'' - m_\nu' m_\nu''$$
$$= M_\nu''(M_\nu' - m_\nu') + m_\nu'(M_\nu'' - m_\nu'') \leqq M''(M_\nu' - m_\nu') + M'(M_\nu'' - m_\nu'');$$

hence

$$\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1}) \leqq M''\Sigma(M_\nu' - m_\nu')(x_\nu - x_{\nu-1}) + M'\Sigma(M_\nu'' - m_\nu'')(x_\nu - x_{\nu-1}).$$

$f_1(x)$ and $f_2(x)$ being integrable, both sums to the right may be made as small as we please by taking the subdivisions sufficiently small, and consequently the same is true of the sum to the left, i. e., $f(x)$ is integrable. Second, if $f_1(x)$ or $f_2(x)$ assumes negative values in (a, b), the product of the non-negative functions $f_1(x) - m'$ and $f_2(x) - m''$ is integrable, and by (17) and (18), the same is the case for

$$f_1(x)f_2(x) = (f_1(x) - m')(f_2(x) - m'') + m''f_1(x) + m'f_2(x) - m'm''.$$

Theorem VI. *When $f(x)$ is integrable in (a, b), and for any x between a and b we write*

$$F(x) = \int_a^x f(t)dt, \qquad x \neq a; \qquad F(a) = 0,$$

then $F(x)$ is continuous in (a, b), and at any point where $f(x)$ is continuous, we have

$$\frac{dF(x)}{dx} = f(x).$$

By (15) we have, when both x and $x + h$ are in (a, b),

$$F(x + h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt,$$

and if M and m are the upper and lower bounds of $f(t)$ in $(x, x + h)$, (12) gives

$$mh \leqq F(x + h) - F(x) \leqq Mh$$

(if h is negative, the inequalities must be reversed by (16)), which proves the continuity of $F(x)$. When $f(x)$ is continuous at x, m and M tend toward the same limit $f(x)$ as $h \to 0$, and consequently

$$f(x) = \lim_{h\to 0} \frac{F(x + h) - F(x)}{h} = \frac{dF(x)}{dx}.$$

Theorem VII. *When $f(x)$ satisfies the conditions of theorem III, and*

there exists a function $F_1(x)$ continuous in (a, b) and the derivative of which equals $f(x)$ at every point where $f(x)$ is continuous, then

$$F_1(x) - F_1(a) = \int_a^x f(t)dt$$

for any x in (a, b).

Let $F(x)$ be the function defined in theorem VI; then $F_1(x)$ and $F(x)$ have the same derivative $f(x)$ in any subinterval of (a, b) in which $f(x)$ is continuous, and by the theorem of the mean in the differential calculus, $F_1(x) = F(x) +$ const. in the subinterval considered. Now let (x_1, x_2) and (x_3, x_4) be two subintervals of continuity for $f(x)$, separated by the subinterval (x_2, x_3) in the interior of which $f(x)$ becomes discontinuous. Then we have $F_1(x) = F(x) + C$ in (x_1, x_2) and $F_1(x) = F(x) + C'$ in (x_3, x_4), and consequently

$$C' - C = (F_1(x_3) - F_1(x_2)) - (F(x_3) - F(x_2)).$$

By the conditions of theorem III, $x_3 - x_2$ may be made as small as we please; since $F_1(x)$ is continuous in (a, b) by hypothesis and $F(x)$ by theorem VI, the expression to the right may be made as small as we please by taking $x_3 - x_2$ sufficiently small, and $C' - C$, being independent of x_2 and x_3, therefore equals zero. Consequently

$$F_1(x) = F(x) + C$$

throughout the interval (a, b), and since $F(a) = 0$, we have $C = F_1(a)$ and the theorem is proved.

THEOREM VIII. *When $f(x)$, $f_1(x)$ and $f_2(x)$ are bounded and integrable in (a, b), then*

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \Sigma f(\xi_\nu)(x_\nu - x_{\nu-1}), \tag{19}$$

$$\int_a^b f_1(x)f_2(x)dx = \lim_{\delta \to 0} \Sigma f_1(\xi_\nu')f_2(\xi_\nu'')(x_\nu - x_{\nu-1}), \tag{20}$$

where ξ_ν, ξ_ν' and ξ_ν'' are any points in $(x_{\nu-1}, x_\nu)$, and $x_\nu - x_{\nu-1} < \delta$ for all ν. When ξ_ν is a point of discontinuity of $f(x)$, we may take as the value of $f(\xi_\nu)$ any number between m and M.

In (19), the sum to the right lies between s and S, and consequently tends toward the same limit $\int_a^b f(x)dx$ as these. In (20), we have

$$\Sigma f_1(\xi_\nu')f_2(\xi_\nu'')(x_\nu - x_{\nu-1})$$
$$= \Sigma f_1(\xi_\nu')f_2(\xi_\nu')(x_\nu - x_{\nu-1}) + \Sigma f_1(\xi_\nu')(f_2(\xi_\nu'') - f_2(\xi_\nu'))(x_\nu - x_{\nu-1});$$

the first sum to the right tends toward $\int_a^b f_1(x)f_2(x)dx$ as $\delta \to 0$ by theorem

V and (19), while the absolute value of the second sum does not exceed

$$\Sigma \,|\, f_1(\xi_\nu') \,|\, \cdot \,|\, f_2(\xi_\nu'') - f_2(\xi_\nu') \,|\, (x_\nu - x_{\nu-1})$$
$$\leqq \Sigma \,|\, f_1(\xi_\nu') \,|\, (M_\nu'' - m_\nu'')(x_\nu - x_{\nu-1}) < (\,|\, M' \,|\, + \,|\, m' \,|\,)\Sigma(M_\nu'' - m_\nu'')(x_\nu - x_{\nu-1})$$

and $f_2(x)$ being integrable, the last sum approaches zero as $\delta \to 0$.

THEOREM IX. **Integration by parts.** *Let $u(x)$ and $v(x)$ be continuous functions of x such that bounded functions $u'(x)$ and $v'(x)$ exist in (a, b) and are continuous and equal to the derivatives of $u(x)$ and $v(x)$ except possibly on a pointset which may be enclosed in the interior of a finite number of subintervals of arbitrarily small total length; then*

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx. \tag{21}$$

Except on the pointset specified, we have

$$uv' + vu' = \frac{d(uv)}{dx},$$

and the integral $\int_a^b (uv' + vu')dx$, which exists by theorem V and (18), equals $u(b)v(b) - u(a)v(a)$ by theorem VII, whence (21) follows by applying (18).

THEOREM X. **Integration by substitution.** *Let $f(x)$ be bounded and integrable in (a, b), and $\varphi(t)$ a function such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$, and $\varphi'(t)$ is continuous and different from zero in (α, β); then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt. \tag{22}$$

Supposing that $\varphi'(t) > 0$ in (α, β), there exists a $k > 0$ such that $\varphi'(t) \geqq k$ in (α, β). Subdivide (α, β) at the points $\alpha = t_0, t_1, \cdots, t_{n-1}, t_n = \beta$, and write $x_\nu = \varphi(t_\nu)$. Let M_ν and m_ν be the upper and lower bounds of $f(\varphi(t))$ in $(t_{\nu-1}, t_\nu)$ or, which is the same, the upper and lower bounds of $f(x)$ in $(x_{\nu-1}, x_\nu)$. Then, by the mean value theorem,

$$x_\nu - x_{\nu-1} = \varphi'(t_\nu')(t_\nu - t_{\nu-1}) > k(t_\nu - t_{\nu-1}) > 0,$$

where t' is some value in $(t_{\nu-1}, t_\nu)$, so that

$$0 < \Sigma(M_\nu - m_\nu)(t_\nu - t_{\nu-1}) < \frac{1}{k}\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1}),$$

and $f(x)$ being integrable in (a, b), the sum to the right approaches zero when all $x_\nu - x_{\nu-1}$ tend toward zero, which is evidently the case when all

$t_\nu - t_{\nu-1}$ tend toward zero, and the above inequality shows that $f(\varphi(t))$ is integrable in (α, β). The same is true when $\varphi'(t) \leqq -k$ in (α, β), the above inequality being then reversed. Since $\varphi'(t)$ is integrable, the product $f(\varphi(t))\varphi'(t)$ is integrable by theorem V, and for t_ν'' in $(t_{\nu-1}, t_\nu)$, $x_\nu' = \varphi(t_\nu'')$, we have

$$\Sigma f(x_\nu')(x_\nu - x_{\nu-1}) = \Sigma f(\varphi(t_\nu''))\varphi'(t_\nu')(t_\nu - t_{\nu-1}).$$

Passing to the limit and using theorem VIII, our proposition follows.

4. Double and repeated integrals. Let $f(x, y)$ be bounded in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and subdivide this rectangle into the mn rectangles $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, where

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$$

and

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d.$$

Denote by M and m the upper and lower bounds of $f(x, y)$ in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and by $M_{\mu\nu}$ and $m_{\mu\nu}$ the corresponding bounds in the rectangle $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, and consider the two sums

$$S = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} M_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}),$$

$$s = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} m_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}).$$

Since evidently $M \geqq M_{\mu\nu} \geqq m_{\mu\nu} \geqq m$, we have

$$M(b - a)(d - c) \geqq S \geqq s \geqq m(b - a)(d - c), \tag{23}$$

so that for all possible modes of subdivision of the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, S and s are bounded above and below; hence there exist an upper bound j of s and a lower bound J of S. In exactly the same way as we proved theorem II, we find

THEOREM XI. *When the number of subrectangles is increased indefinitely in such a manner that the sides $x_\mu - x_{\mu-1}$ and $y_\nu - y_{\nu-1}$ of each tend toward zero, the sums S and s tend toward the limits J and j respectively.* The numbers J and j are called the upper and lower integrals of $f(x, y)$ over the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and are denoted by

$$J = \overline{\int_a^b \int_c^d} f(x, y)dxdy, \qquad j = \underline{\int_a^b \int_c^d} f(x, y)dxdy.$$

When $f(x, y)$ is bounded in the rectangle, and the upper and lower integrals of $f(x, y)$ over the rectangle are equal, the function is said to be *integrable* in the rectangle, and the common value of J and j is called the (double)

integral of $f(x, y)$ and denoted by

$$\int_a^b \int_c^d f(x, y)\,dx\,dy.$$

The necessary and sufficient condition for integrability is obviously that the difference (which is $\geqq 0$ by (23))

$$S - s = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (M_{\mu\nu} - m_{\mu\nu})(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1})$$

shall tend toward zero for some particular mode of subdivision of the rectangle when the number of subrectangles is increased indefinitely in such a manner that the lengths of all their sides approach zero.

The following theorem is proved in exactly the same way as theorem III:

THEOREM XII. *A bounded function $f(x, y)$ is integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, when it is possible to find a finite number of subrectangles, the total area of which is as small as we please, and such that every point of discontinuity of $f(x, y)$ is interior to one of these subrectangles.*

For any value of x in (a, b), $f(x, y)$ is bounded when $c \leqq y \leqq d$, and consequently the upper and lower integrals $\overline{\int_c^d} f(x, y)dy$ and $\underline{\int_c^d} f(x, y)dy$ exist. Denote by

$$F(x) = \overline{\underline{\int_c^d}} f(x, y)dy$$

any number such that

$$\overline{\int_c^d} f(x, y)dy \geqq \overline{\underline{\int_c^d}} f(x, y)dy \geqq \underline{\int_c^d} f(x, y)dy,$$

so that, in particular, $F(x) = \int_c^d f(x, y)dy$ for every value of x for which $f(x, y)$ is integrable in respect to y between the limits c and d. Then the integral (which we suppose for the moment to exist) $\int_a^b F(x)dx$ is called the *repeated integral* of $f(x, y)$ in respect to y and x and is denoted by

$$\int_a^b dx \int_c^d f(x, y)dy.$$

Similarly, we may define the repeated integral in respect to x and y,

$$\int_c^d dy \int_a^b f(x, y)dx = \int_c^d G(y)dy,$$

where $G(y) = \overline{\underline{\int_a^b}} f(x, y)dx$. We then have

Theorem XIII. *When $f(x, y)$ is bounded and integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, then both repeated integrals of $f(x, y)$ exist and are equal to the double integral:*

$$\int_a^b dx \int_c^d f(x, y)dy = \int_c^d dy \int_a^b f(x, y)dx = \int_a^b \int_c^d f(x, y)dxdy.$$

Since $m_{\mu\nu} \leqq f(x, y) \leqq M_{\mu\nu}$ for $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, it follows from (9) that

$$m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq \underline{\int}_{y_{\nu-1}}^{y_\nu} f(x, y)dy \leqq \overline{\int}_{y_{\nu-1}}^{y_\nu} f(x, y)dy \leqq M_{\mu\nu}(y_\nu - y_{\nu-1});$$

adding these inequalities for $\nu = 1, 2, \cdots, n$ and applying (8), we find

$$\sum_{\nu=1}^{n} m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq \underline{\int}_c^d f(x, y)dy \leqq \overline{\int}_c^d f(x, y)dy \leqq \sum_{\nu=1}^{n} M_{\mu\nu}(y_\nu - y_{\nu-1})$$

for any x in $(x_{\mu-1}, x_\mu)$, and consequently the lower bound m_μ and the upper bound M_μ of $F(x) = \overline{\underline{\int}}_c^d f(x, y)dy$ for $x_{\mu-1} \leqq x \leqq x_\mu$ satisfy the inequalities

$$\sum_{\nu=1}^{n} m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq m_\mu \leqq M_\mu \leqq \sum_{\nu=1}^{n} M_{\mu\nu}(y_\nu - y_{\nu-1}).$$

Multiplication by $x_\mu - x_{\mu-1}$ and summation in respect to μ now gives

$$\begin{aligned}\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} m_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}) &\leqq \sum_{\mu=1}^{m} m_\mu(x_\mu - x_{\mu-1}) \\ &\leqq \sum_{\mu=1}^{m} M_\mu(x_\mu - x_{\mu-1}) \\ &\leqq \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} M_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}).\end{aligned}$$

But as m and n tend toward infinity while all $x_\mu - x_{\mu-1}$ and $y_\nu - y_{\nu-1}$ tend toward zero, the two outside members of this inequality approach the same limit $\int_a^b \int_c^d f(x, y)dxdy$; consequently the two inner members must also approach this same limit, that is, $\int_a^b F(x)dx$ exists and equals the double integral. The same proof obviously applies also to the other repeated integral.

Theorem XIV. *When $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial y}$ are bounded for x in (a, b) and y in (c, d) and*

$$\int_a^b \int_c^y \frac{\partial f(x, t)}{\partial t}\, dxdt$$

exists for any y in (c, d),

$$\int_c^y \frac{\partial f(x, t)}{\partial t}\,dt$$

exists for any x in (a, b) and any y in (c, d), and finally $\int_a^b f(x, c)dx$ exists, then

$$\frac{\partial}{\partial y}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f(x, y)}{\partial y}\,dx$$

for every value of y for which the last integral is a continuous function of y.

It may be noted that all the conditions of the theorem are satisfied when $f(x, y)$ and $\partial f(x, y)/\partial y$ are continuous for x in (a, b) and y in (c, d).

For the proof we observe that by theorem XIII

$$(24)\qquad \int_a^b\int_c^y \frac{\partial f(x, t)}{\partial t}\,dxdt = \int_a^b dx\int_c^y \frac{\partial f(x, t)}{\partial t}\,dt = \int_c^y dt\int_a^b \frac{\partial f(x, t)}{\partial t}\,dx,$$

and by theorem VII

$$\int_c^y \frac{\partial f(x, t)}{\partial t}\,dt = f(x, y) - f(x, c);$$

hence, by theorem XIII, $\int_a^b (f(x, y) - f(x, c))dx$ exists, and from the existence of $\int_a^b f(x, c)dx$ and (18) we conclude that

$$\int_a^b f(x, y)dx = \int_a^b f(x, c)dx + \int_a^b (f(x, y) - f(x, c))dx.$$

Consequently (24) gives

$$\int_a^b f(x, y)dx - \int_a^b f(x, c)dx = \int_c^y dt\int_a^b \frac{\partial f(x, t)}{\partial t}\,dx,$$

and differentiating in respect to y, theorem VI gives

$$\frac{\partial}{\partial y}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f(x, y)}{\partial y}\,dx$$

for any value of y for which the integral to the right is a continuous function of y.

The following proposition is proved in exactly the same way as theorem IV:

THEOREM XV. *When $f(x, y)$ is bounded and integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, the same is true of $|f(x, y)|$.*

Formulas (13) and (14) are also immediately extended to double

integrals:

$$(25)\qquad \int_a^b\int_c^d m(x, y)dxdy \leqq \int_a^b\int_c^d f(x, y)dxdy \leqq \int_a^b\int_c^d M(x, y)dxdy$$

when the three bounded and integrable functions $f(x, y)$, $m(x, y)$ and $M(x, y)$ satisfy the inequalities $m(x, y) \leqq f(x, y) \leqq M(x, y)$ in the rectangle of integration; and for $m(x, y) = -|f(x, y)|$, $M(x, y) = |f(x, y)|$,

$$(26)\qquad \left|\int_a^b\int_c^d f(x, y)dxdy\right| \leqq \int_a^b\int_c^d |f(x, y)|\, dxdy.$$

Also (17) and (18) extend immediately to double integrals:

$$(27)\qquad \int_a^b\int_c^d kf(x, y)dxdy = k\int_a^b\int_c^d f(x, y)dxdy,$$

$$\begin{aligned}&\int_b^b\int_d^d (f_1(x, y) + f_2(x, y) + \cdots + f_n(x, y))dxdy\\ (28)\qquad &= \int_a^b\int_c^d f_1(x, y)dxdy + \int_a^b\int_c^d f_2(x, y)dxdy + \cdots\\ &\qquad + \int_a^b\int_c^d f_n(x, y)dxdy.\end{aligned}$$

In (25)–(28), any double integral may evidently be replaced by either of the repeated integrals. Finally, theorem VII extends immediately to double and repeated integrals.

5. **Infinite integrals.** In the preceding paragraphs, we have assumed a and b finite and $f(x)$ bounded in (a, b). When one of these conditions is not fulfilled, we may define, by a limiting process, what is called an infinite integral (the term is chosen in analogy to "infinite series").

First assume $f(x)$ to be bounded and integrable in (a, b) for any $b > a$. The integral between the limits a and ∞ is then defined as

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx,$$

whenever this limit exists. Similarly we may define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx,\quad \int_{-\infty}^\infty f(x)dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)dx.$$

Next let us assume that $f(x)$ is bounded and integrable in $(a, b - \epsilon)$ for an arbitrarily small positive ϵ, but tends toward $+\infty$ or $-\infty$ (or oscillates between these bounds) as $x \to b$. The integral between the limits a and b is then defined as

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)dx$$

whenever this limit exists. Further, when $f(x)$ ceases to be bounded as $x \to a$, we define

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)dx,$$

and when c is a point between a and b in the neighborhood of which $f(x)$ is not bounded

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon' \to 0} \int_{c+\epsilon'}^b f(x)dx.$$

Similarly we may define infinite repeated integrals; for instance, when $f(x, y)$ is bounded and integrable for $a + \epsilon \leqq x \leqq b$, $c + \epsilon' \leqq y \leqq d$, where ϵ and ϵ' are as small, b and d as large as we please, we have the definitions

$$\int_a^\infty dx \int_c^\infty f(x, y)dy = \lim_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_{a+\epsilon}^b dx \left[\lim_{\substack{\epsilon' \to 0 \\ d \to \infty}} \int_{c+\epsilon'}^d f(x, y)dy \right],$$

$$\int_c^\infty dy \int_a^\infty f(x, y)dx = \lim_{\substack{\epsilon' \to 0 \\ d \to \infty}} \int_{c+\epsilon'}^d dy \left[\lim_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_{a+\epsilon}^b f(x, y)dx \right].$$

In the following, we shall consider only integrals of the type $\int_a^\infty f(x)dx$ where $f(x)$ may cease to be bounded at $x = a$, and the corresponding case for repeated integrals. The theorems we shall prove hold, however, in the other cases; the necessary modifications of the proofs are obvious.

A convenient means of establishing the existence of an infinite integral consists in comparing it to another with positive integrand.

THEOREM XVI. *Let $f(x)$ be bounded and integrable when $a + \epsilon \leqq x \leqq b$, where ϵ is as small and b as large as we please, and let $|f(x)| \leqq M(x)$ in this interval. If $\int_{a+\epsilon}^b M(x)dx$ exists, and is bounded when $\epsilon \to 0$ and $b \to \infty$, then $\int_a^\infty f(x)dx$ exists, and*

$$\left| \int_a^\infty f(x)dx \right| \leqq \int_a^\infty M(x)dx.$$

Since $M(x) \geqq 0$, $\int_{a+\epsilon}^b M(x)dx$ increases (or remains constant) when ϵ decreases or b increases, and being bounded, this integral must therefore approach a definite limit as $\epsilon \to 0$ and $b \to \infty$, that is, $\int_a^\infty M(x)dx$ exists.

Consequently

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} M(x)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_{b}^{b'} M(x)dx = 0,$$

and by (14) and (12)

$$\left| \int_{a+\epsilon}^{a+\epsilon'} f(x)dx \right| \leqq \int_{a+\epsilon}^{a+\epsilon'} M(x)dx, \qquad \left| \int_{b}^{b'} f(x)dx \right| \leqq \int_{b}^{b'} M(x)dx,$$

so that

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} f(x)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_{b}^{b'} f(x)dx = 0,$$

which are the conditions for the existence of $\int_a^\infty f(x)dx$.

Corollary. *When* $|f(x)| < K(x-a)^{-\alpha}$, *where* $\alpha < 1$, *for* $a < x < x_1$, *and* $|f(x)| < Kx^{-\beta}$, *where* $\beta > 1$, *for* $x > x_2$, *then* $\int_a^\infty f(x)dx$ *exists.*

In fact, when ϵ and ϵ' are so small that $a + \epsilon < a + \epsilon' < x_1$, we have

$$\left| \int_{a+\epsilon}^{a+\epsilon'} f(x)dx \right| \leqq \int_{a+\epsilon}^{a+\epsilon'} K(x-a)^{-\alpha}dx = \frac{K}{1-\alpha}(\epsilon'^{1-\alpha} - \epsilon^{1-\alpha}) \to 0,$$

and for $b' > b > x_2$,

$$\left| \int_{b}^{b'} f(x)dx \right| \leqq \int_{b}^{b'} Kx^{-\beta}dx = \frac{K}{\beta - 1}\left(\frac{1}{b^{\beta-1}} - \frac{1}{b'^{\beta-1}}\right) \to 0.$$

When $\int_a^\infty f(x)dx$ exists, then, by the definition of an infinite integral, $\int_a^{x_1} f(x)dx$ and $\int_{x_{n-1}}^\infty f(x)dx$ also exist, and (15) becomes

$$(29) \qquad \int_a^\infty f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)dx + \int_{x_{n-1}}^\infty f(x)dx$$

for $a < x_1 < x_2 < \cdots < x_{n-1}$. The equation of definition (16) becomes

$$(30) \qquad \int_\infty^a f(x)dx = -\int_a^\infty f(x)dx,$$

and (17) and (18) also extend immediately to the present case:

$$(31) \qquad \int_a^\infty kf(x)dx = k\int_a^\infty f(x)dx,$$

$$(32) \qquad \begin{aligned} &\int_a^\infty (f_1(x) + f_2(x) + \cdots + f_n(x))dx \\ &\qquad = \int_a^\infty f_1(x)dx + \int_a^\infty f_2(x)dx + \cdots + \int_a^\infty f_n(x)dx. \end{aligned}$$

By a passage to the limit in theorem VII, we find

THEOREM XVII. *When $f(x)$ satisfies the conditions of theorem III in $(a+\epsilon, b)$ where ϵ is as small and b as large as we please, and there exists a function $F_1(x)$ continuous in $(a+\epsilon, b)$ and the derivative of which equals $f(x)$ at every point where $f(x)$ is continuous, when finally*

$$\lim_{\epsilon\to 0} F_1(a+\epsilon) = F_1(a)$$

and $\lim_{b\to\infty} F_1(b) = F_1(\infty)$ *both exist, then* $\int_a^\infty f(x)dx$ *exists and*

$$F_1(\infty) - F_1(a) = \int_a^\infty f(x)dx.$$

Similarly, it is seen that when the conditions of theorem IX are fulfilled in $(a+\epsilon, b)$, where ϵ is as small and b as large as we please, and three of the four expressions $\int_a^\infty u(x)v'(x)dx$, $\int_a^\infty v(x)u'(x)dx$, $u(a)v(a) = \lim_{\epsilon\to 0} u(a+\epsilon)v(s+\epsilon)$ and $u(\infty)v(\infty) = \lim_{b\to\infty} u(b)v(b)$ exist, then the fourth exists and

$$\int_a^\infty u(x)v'(x)dx = u(\infty)v(\infty) - u(a)v(a) - \int_a^\infty v(x)u'(x)dx. \tag{33}$$

Finally, it is readily seen how theorem X, on integration by substitution, may be extended to the present case.

A convergent infinite series $u_0 + u_1 + \cdots + u_n + \cdots$ may be written as an infinite integral by making $f(x) = u_n$ for $n \leqq x < n+1$. In the interval $(0, b)$, where $n+1 \leqq b < n+2$, $f(x)$ has the only discontinuities $x = 1, 2, \cdots, n+1$ and is integrable by theorem III, and (15) gives

$$\int_0^b f(x)dx = \sum_{\nu=0}^{n}\int_\nu^{\nu+1} f(x)dx + \int_{n+1}^b f(x)dx = \sum_{\nu=0}^{n} u_\nu + (b-n-1)u_{n+1}.$$

The series being convergent, $u_{n+1}\to 0$ as $n\to\infty$, so that

$$|(b-n-1)u_{n+1}| < |u_{n+1}| \to 0.$$

In the following, the concept of *uniform approach to a limit* (or *uniform convergence*) is of fundamental importance. Let $f(x, \delta, \omega)$ depend on the parameters δ and ω; then $f(x, \delta, \omega)$ tends toward the limit $f(x)$ *uniformly* in (a, b) as $\delta\to 0$ and $\omega\to\infty$ when, for any $\epsilon > 0$ as small as we please, there exist a $\delta_0 = \delta_0(\epsilon) > 0$ and an $\omega_0 = \omega_0(\epsilon) > 0$ independent of x such that for every x in (a, b)

$$|f(x, \delta, \omega) - f(x)| < \epsilon \text{ for } 0 < \delta < \delta_0,\ \omega > \omega_0.$$

In particular, let $f(x, \delta, \omega)$ be independent of δ and make

$$f(x, \omega) = \sum_{\nu=0}^{n} u_\nu(x)$$

for $n \leqq \omega < n + 1$; we then obtain the familiar definition of uniform convergence of a series.

THEOREM XVIII. *When for* $a \leqq x \leqq b$, $f(x, \delta, \omega)$ *is bounded and integrable in respect to* x *and tends uniformly toward* $f(x)$ *as* $\delta \to 0$ *and* $\omega \to \infty$, *then* $\int_a^b f(x)dx$ *exists and*

$$\lim_{\substack{\delta \to 0 \\ \omega \to \infty}} \int_a^b f(x, \delta, \omega)dx = \int_a^b f(x)dx.$$

For any $\epsilon > 0$, there exist, by hypothesis, a $\delta_0 = \delta_0(\epsilon)$ and an $\omega_0 = \omega_0(\epsilon)$ such that $f(x, \delta, \omega) - \epsilon \leqq f(x) \leqq f(x, \delta, \omega) + \epsilon$ in (a, b) for $0 < \delta < \delta_0$, $\omega > \omega_0$. Hence $f(x)$ is bounded in (a, b) so that its upper and lower integrals exist; the application of (11) and (9) to the above inequality gives

$$\overline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x, \delta, \omega)dx + \epsilon(b - a),$$

$$\underline{\int_a^b} f(x)dx \geqq \underline{\int_a^b} f(x, \delta, \omega)dx - \epsilon(b - a).$$

Since $f(x, \delta, \omega)$ is integrable, its upper and lower integrals are equal, whence $0 \leqq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leqq 2\epsilon(b - a)$; but $f(x)$ being independent of ϵ, it follows that $\int_a^b f(x)dx$ exists, and the preceding inequalities become

$$-\epsilon(b - a) \leqq \int_a^b f(x)dx - \int_a^b f(x, \delta, \omega)dx \leqq \epsilon(b - a)$$

for $0 < \delta < \delta_0$, $\omega > \omega_0$, which proves our proposition.

The integral $\int_a^\infty f(x, y)dx$ is said to *exist uniformly* for y in (c, d) when $f(x, y)$ is bounded and integrable in respect to x in $(a + \epsilon, b)$ and the integral $\int_{a+\epsilon}^b f(x, y)dx$ tends toward $\int_a^\infty f(x, y)dx$ uniformly for y in (c, d), that is, when

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} f(x, y)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_b^{b'} f(x, y)dx = 0$$

uniformly for y in (c, d). A convenient test for uniform existence is contained in

THEOREM XIX. *When* $f(x, y)$ *is bounded and integrable in respect to* x *in* $(a + \epsilon, b)$, *while* y *is in* (c, d), *and when, for* $x > a$ *and* y *in* (c, d),

$|f(x, y)| < M(x)$ where $\int_a^\infty M(x)dx$ exists, then $\int_a^\infty f(x, y)dx$ exists uniformly for y in (c, d).

In fact, the proof of theorem XVI then shows that both the integrals

$$\int_{a+\epsilon}^{a+\epsilon'} f(x, y)dx \quad \text{and} \quad \int_b^{b'} f(x, y)dx$$

approach zero uniformly as $\epsilon, \epsilon' \to 0$ and $b, b' \to \infty$.

We furthermore have

THEOREM XX. *When $f(x, y)$ is bounded and doubly integrable for $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, and $\int_a^\infty f(x, y)dx$ exists uniformly for y in (c, d), then*

$$\int_c^d dy \int_a^\infty f(x, y)dx = \int_a^\infty dx \int_c^d f(x, y)dy.$$

By theorem XIII, we have

$$\int_c^d dy \int_{a+\epsilon}^b f(x, y)dx = \int_{a+\epsilon}^b dx \int_c^d f(x, y)dy,$$

and since $\int_a^\infty f(x, y)dx$ exists uniformly for $c \leqq y \leqq d$, theorem XVIII shows that $\int_c^d dy \int_a^\infty f(x, y)dx$ exists and equals $\lim\limits_{\substack{\epsilon\to 0\\ b\to\infty}} \int_c^d dy \int_{a+\epsilon}^b f(x, y)dx$. The preceding equation then shows that $\lim\limits_{\substack{\epsilon\to 0\\ b\to\infty}} \int_{a+\epsilon}^b dx \int_c^d f(x, y)dy$ exists and equals $\int_c^d dy \int_a^\infty f(x, y)dx$. But the last limit, by the definition of an infinite integral, equals $\int_a^\infty dx \int_c^d f(x, y)dy$, whence our theorem.

Making $f(x, y) = u_n(y)$ for $n \leqq x < n + 1$, where all $u_n(y)$ are bounded and integrable for $c \leqq y \leqq d$, the uniform existence of $\int_0^\infty f(x, y)dy$ is equivalent to the uniform convergence of $\sum_0^\infty u_n(y)$ in (c, d), whence writing x, a, b for y, c, d, we find the

COROLLARY. *When $\sum_0^\infty u_n(x)$ converges uniformly in (a, b), every $u_n(x)$ being bounded and integrable, then*

$$\int_a^b \sum_0^\infty u_n(x)dx = \sum_0^\infty \int_a^b u_n(x)dx.$$

THEOREM XXI. *When* $f(x, y)$ *and* $\partial f(x, y)/\partial y$ *are continuous for* $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, $\int_a^\infty f(x, c)dx$ *exists and* $\int_a^\infty (\partial f(x, y)/\partial y)dx$ *exists uniformly for* y *in* (c, d), *then* $\int_a^\infty f(x, y)dx$ *exists uniformly for* y *in* (c, d), *and*

$$\frac{\partial}{\partial y}\int_a^\infty f(x, y)dx = \int_a^\infty \frac{\partial f(x, y)}{\partial y}dx.$$

All conditions of theorem XIV being fulfilled for x in $(a + \epsilon, b)$ and y in (c, d), we have

$$\int_{a+\epsilon}^b f(x, y)dx = \int_{a+\epsilon}^b f(x, c)dx + \int_c^y dt\int_{a+\epsilon}^b \frac{\partial f(x, t)}{\partial t}dx.$$

Since the first integral to the right is independent of y and tends toward $\int_a^\infty f(x, c)dx$ as $\epsilon \to 0$, $b \to \infty$, and since $\int_c^y dt\int_a^\infty (\partial f(x, t)/\partial t)dx$ exists uniformly for y in (c, d) by the hypothesis on $\int_a^\infty (\partial f(x, y)/\partial y)dx$ and theorem XVIII, it follows that $\int_a^\infty f(x, y)dx$ exists uniformly for y in (c, d) and

$$\int_a^\infty f(x, y)dx = \int_a^\infty f(x, c)dx + \int_c^y dt\int_a^\infty \frac{\partial f(x, t)}{\partial t}dx.$$

From the continuity of $\partial f(x, y)/\partial y$ for $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, and the uniform existence of $\int_a^\infty (\partial f(x, y)/\partial y)dx$ for y in (c, d), it follows at once that the latter integral is a continuous function of y in (c, d). Consequently, differentiating the last equation in respect to y and applying theorem VI, we obtain our proposition.

Making $f(x, y) = u_n(y)$ for $n \leqq x < n + 1$, and replacing y, c, d by x, a, b, we have the

COROLLARY. *When* $\sum_0^\infty (du_n(x)/dx)$ *converges uniformly in* (a, b), *every* $(du_n(x)/dx)$ *being continuous, and* $\sum_0^\infty u_n(a)$ *converges, then* $\sum_0^\infty u_n(x)$ *converges uniformly in* (a, b), *and*

$$\frac{d}{dx}\sum_0^\infty u_n(x) = \sum_0^\infty \frac{du_n(x)}{dx}.$$

Regarding infinite repeated integrals, we shall now prove

THEOREM XXII. *When* $f(x, y)$ *is bounded and integrable for* $a + \epsilon \leqq x \leqq b$, $c + \epsilon' \leqq y \leqq d$, *where* ϵ, ϵ' *are as small and* b, d *as large as we please,*

and ONE *of the infinite repeated integrals* $\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$ *and* $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$ *exists, then both the infinite repeated integrals of* $f(x, y)$ *exist and are equal:*

$$\int_a^\infty dx \int_c^\infty f(x, y)dy = \int_c^\infty dy \int_a^\infty f(x, y)dx.$$

From the existence of $\int_{a+\epsilon}^{b} \int_{c+\epsilon'}^{d} f(x, y)dxdy$ we infer, by theorem XV, that of $\int_{a+\epsilon}^{b} \int_{c+\epsilon'}^{d} |f(x, y)|\, dxdy$ and hence by theorem XIII that

$$\int_{a+\epsilon}^{b} dx \int_{c+\epsilon'}^{d} |f(x, y)|\, dy = \int_{c+\epsilon'}^{d} dy \int_{a+\epsilon}^{b} |f(x, y)|\, dx.$$

When in either of these repeated integrals ϵ and ϵ' are decreased, b and d increased, the integral cannot decrease since $|f(x, y)|$ is never negative. Consequently assuming the existence of $\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$, we have $\int_{a+\epsilon}^{b} dx \int_{c+\epsilon'}^{d} |f(x, y)|\, dy \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$, whence

$$\int_{c+\epsilon'}^{d} dy \int_{a+\epsilon}^{b} |f(x, y)|\, dx \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy,$$

so that the integral to the left is bounded for all values of ϵ, ϵ', b and d considered. Since this integral does not decrease as ϵ, ϵ' decrease to zero and b, d increase indefinitely, it therefore approaches a limit not exceeding the right member, that is, $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$ exists and

$$\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy.$$

Having established the existence of $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$, we apply to it the same argument as was applied to the other infinite repeated integral, and thereby establish the inequality

$$\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy \leqq \int_c^\infty dy \int_a^\infty |f(x, y)|\, dx.$$

Combining the two inequalities, we find

$$\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy = \int_c^\infty dy \int_a^\infty |f(x, y)|\, dx.$$

Furthermore, the integral $\int_{a+\epsilon}^{b}\int_{c+\epsilon'}^{d}(|f(x,y)| - f(x,y))dxdy$ exists, being the difference of the double integrals of $|f(x,y)|$ and $f(x,y)$, and from $0 \leqq |f(x,y)| - f(x,y) \leqq 2|f(x,y)|$ we see that

$$0 \leqq \int_{a+\epsilon}^{b} dx \int_{c+\epsilon'}^{d}(|f(x,y)| - f(x,y))dy \leqq 2\int_{a}^{\infty} dx \int_{c}^{\infty} |f(x,y)|\, dy.$$

Since the integral to the left cannot decrease as ϵ, ϵ' decrease to zero and b, d increase indefinitely, it follows that $\int_{a}^{\infty} dx \int_{c}^{\infty}(|f(x,y)| - f(x,y))dy$ exists; we may now show exactly as before that the other infinite repeated integral exists and that

$$\int_{a}^{\infty} dx \int_{c}^{\infty}(|f(x,y)| - f(x,y))dy = \int_{c}^{\infty} dy \int_{a}^{\infty}(|f(x,y)| - f(x,y))dx.$$

Our theorem now follows by subtracting this equation from that relative to $|f(x,y)|$.

Replacing $f(x,y)$ by $u_n(y)$ for $n \leqq x < n+1$ and writing x, a for y, c we have the

COROLLARY. *When every $u_n(x)$ is finite and integrable in $(a+\epsilon, b)$ and either of the expressions $\int_{a}^{\infty}\sum_{0}^{\infty}|u_n(x)|\,dx$ and $\sum_{0}^{\infty}\int_{a}^{\infty}|u_n(x)|\,dx$ exists, then both the corresponding expressions in $u_n(x)$ exist, and*

$$\int_{a}^{\infty}\sum_{0}^{\infty} u_n(x)dx = \sum_{0}^{\infty}\int_{a}^{\infty} u_n(x)dx.$$

6. **Integrals of complex functions of a real variable.** In the following two chapters, we shall make extensive use of such integrals, which are defined in the following manner: let $f(x) = g_1(x) + ig_2(x)$, where $g_1(x)$ and $g_2(x)$ are real functions of x, bounded and integrable for $a \leqq x \leqq b$. Then, by definition,

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} g_1(x)dx + i\int_{a}^{b} g_2(x)dx,$$

and the double and repeated integrals of $f(x,y) = g_1(x,y) + ig_2(x,y)$, as well as the various kinds of infinite integrals, are defined in the same manner by decomposition into their real and imaginary parts.

Theorems II and XI, as well as formulas (7)–(13), apply to real functions only; but all the remaining theorems and formulas may be extended to the present case. In fact, the proof of theorem I subsists unchanged, since it makes no use of the reality of $f(x_1, \cdots, x_n)$. Theorem III is

immediately extended by decomposition of $f(x)$ into its real and complex parts. In theorem IV, let M_ν, m_ν, $M_{1\nu}$, $m_{1\nu}$, $M_{2\nu}$, $m_{2\nu}$ be the upper and lower bounds of $|f(x)|$, $|g_1(x)|$ and $|g_2(x)|$ in $(x_{\nu-1}, x_\nu)$, and x', x'' any two points in this interval. We have

$$\begin{aligned}|f(x')| - |f(x'')| \leqq |f(x') - f(x'')|& \\ &= |g_1(x') - g_1(x'') + i(g_2(x') - g_2(x''))| \\ &\leqq |g_1(x') - g_1(x'')| + |g_2(x') - g_2(x'')| \\ &\leqq (M_{1\nu} - m_{1\nu}) + (M_{2\nu} - m_{2\nu}),\end{aligned}$$

and since for suitably chosen values of x' and x'' $|f(x')|$ will differ from M_ν, and $|f(x'')|$ from m_ν, as little as we please, it follows that

$$M_\nu - m_\nu \leqq (M_{1\nu} - m_{1\nu}) + (M_{2\nu} - m_{2\nu}),$$

and consequently the $S - s$ relative to $|f(x)|$ may be made as small as we please, since it is less than the sum of the $S - s$ relative to $|g_1(x)|$ and that relative to $|g_2(x)|$, both of which may be made as small as we please by theorem IV. Theorems V–X, XII–XIV are immediately extended by decomposition (it being observed that in X $\varphi(t)$ is a real function of the real variable t), XV is extended in the same way as IV, and the proof of XVI remains unchanged, since (14) subsists in the complex case. In fact, with the notations of theorem VIII, we have, since $x_\nu - x_{\nu-1} > 0$, for $f(x)$ complex

$$|\Sigma f(\xi_\nu)(x_\nu - x_{\nu-1})| \leqq \Sigma |f(\xi_\nu)| (x_\nu - x_{\nu-1});$$

as we have seen that theorem VIII extends to the complex case, it follows that for $\delta \to 0$, the left side approaches $\left|\int_a^b f(x)dx\right|$, and the right, $\int_a^b |f(x)| dx$, whence we obtain (14). In exactly the same way, we extend (26), while (15), (16), (17), where k may now be a complex constant, (18), (27), (28), (32), (33) are extended immediately by decomposition. Theorems XVII and XVIII extend by decomposition, the proof of XIX remains unchanged, XX and XXI (with corollaries) are again extended by decomposition, while in XXII (and corollary) we first conclude from $|g_1(x, y)| \leqq |f(x, y)|$ and $|g_2(x, y)| \leqq |f(x, y)|$ that the same infinite repeated integrals of $|g_1(x, y)|$ and $|g_2(x, y)|$ exist as that of $|f(x, y)|$ included in the hypothesis; the extension is then finished by decomposition.

We shall finally prove the following

THEOREM XXIII. *When, for an $\epsilon > 0$ as small and a b as large as we please, the function $f(s, t)$ of two complex variables s and t is holomorphic for*

$a + \epsilon \leqq t \leqq b$ and $|s - s_0| \leqq r$, where r is independent of ϵ and b, and when finally $\int_a^\infty f(s, t)dt$ converges uniformly for $|s - s_0| < r$, then this integral defines a function of s which is holomorphic for $|s - s_0| < r$.

Write $s - s_0 = x + iy$ and $f(s, t) = g_1(x, y, t) + ig_2(x, y, t)$, and denote by P the pointset defined by $a + \epsilon \leqq t \leqq b$, $x^2 + y^2 \leqq r^2$; since $f(s, t)$ is holomorphic on P, g_1 and g_2 are continuous on P and this pointset being closed, g_1 and g_2 are uniformly continuous on P (theorem I). Since g_1 and g_2 are continuous, we may take the whole interval $(a + \epsilon, b)$ as Δ'' in theorem III, and on account of the uniform continuity on P, we may choose, for any $\epsilon' > 0$, a δ *independent* of x and y such that S and s differ by less than ϵ' when all subdivisions of $(a + \epsilon, b)$ are less than δ. Consequently, as $\delta \to 0$, the sum

$$\Sigma f(s, \tau_\nu)(t_\nu - t_{\nu-1}) = \Sigma g_1(x, y, \tau_\nu)(t_\nu - t_{\nu-1}) + i\Sigma g_2(x, y, \tau_\nu)(t_\nu - t_{\nu-1}),$$

formed according to theorem VIII, tends toward $\int_{a+\epsilon}^b f(s, t)dt$ uniformly for $|s - s_0| \leqq r$. But each $f(s, \tau_\nu)$ is expansible in a power series in $s - s_0$ convergent for $|s - s_0| \leqq r$, and therefore also $\Sigma f(s, \tau_\nu)(t_\nu - t_{\nu-1})$ which contains a finite number of such terms. Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$, and let

$$\Phi_n(s; \epsilon, b) = \Sigma f(s, \tau_\nu)(t_\nu - t_{\nu-1})$$

for a certain mode of subdivision where all $t_\nu - t_{\nu-1} < \delta_n$; since $\Phi_n(s; \epsilon, b)$ is a power series in $s - s_0$ convergent for $|s - s_0| \leqq r$ and since $\Phi_n(s; \epsilon, b)$ tends uniformly toward $\int_{a+\epsilon}^b f(s, t)dt$, we conclude from Weierstrass' theorem on sequences of analytic functions that $\int_{a+\epsilon}^b f(s, t)dt$ is expansible in a power series in $s - s_0$ convergent for $|s - s_0| \leqq r$. Now choose sequences $\epsilon_1, \epsilon_2, \cdots$ and $b_1, b_2, \cdots$ such that $\epsilon_n \to 0$ and $b_n \to \infty$ as $n \to \infty$; since $\int_a^\infty f(s, t)dt$ converges uniformly for $|s - s_0| \leqq r$, the expressions $\int_{a+\epsilon_n}^{b_n} f(s, t)dt$, which are power series in $s - s_0$ convergent for $|s - s_0| \leqq r$, converge uniformly toward the limit $\int_a^\infty f(s, t)dt$ which is therefore, by a second application of Weierstrass' theorem, a power series convergent for $|s - s_0| < r$.

7. **Fourier series.** Let $f(x)$ be bounded and integrable for $\epsilon \leqq x \leqq 1 - \epsilon$, where ϵ is as small as we please, and suppose that $\int_0^1 |f(x)| dx$ exists.

Since $|f(x) \sin 2n\pi x| \leqq |f(x)|$ and $|f(x) \cos 2n\pi x| \leqq |f(x)|$, it follows from theorem XVI that the integrals

$$a_n = 2\int_0^1 f(x) \cos 2n\pi x dx, \qquad b_n = 2\int_0^1 f(x) \sin 2n\pi x dx \tag{34}$$

both exist. Then the Fourier series of $f(x)$ is defined as

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos 2n\pi x + b_n \sin 2n\pi x). \tag{35}$$

In regard to the convergence of this series, we shall need in the following only this very special theorem:

When $f(x)$ is holomorphic for $0 < x < 1$ (but not necessarily at the end points of the interval) and $\int_0^1 |f(x)| dx$ exists, the Fourier series of $f(x)$ converges to the sum $f(x)$ uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, where $\epsilon > 0$ but as small as we please.

Using the integration variable t instead of x in (34), we find

$$\begin{aligned} a_n \cos 2n\pi x + b_n \sin 2n\pi x \\ &= 2\int_0^1 f(t)(\cos 2n\pi t \cos 2n\pi x + \sin 2n\pi t \sin 2n\pi x) dt \\ &= 2\int_0^1 f(t) \cos 2n\pi(t - x) dt, \end{aligned}$$

and since we have[3]

$$1 + 2\sum_{\nu=1}^{n} \cos 2\nu\pi(t - x) = \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)}$$

it follows that

$$\begin{aligned} \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos 2\nu\pi x + b_\nu \sin 2\nu\pi x) \\ &= \int_0^1 f(t) \left[1 + 2\sum_{\nu=1}^{n} \cos 2\nu\pi(t - x)\right] dt \\ &= \int_0^1 f(t) \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)} dt. \end{aligned} \tag{36}$$

[3] Proof: in the finite geometric series

$$\sum_{\nu=0}^{n} e^{(\alpha+2\nu\beta)i} = e^{\alpha i} \cdot \frac{e^{2(n+1)\beta i} - 1}{e^{2\beta i} - 1} = e^{\alpha i} \frac{e^{(2n+1)\beta i} - e^{-\beta i}}{e^{\beta i} - e^{-\beta i}}$$
$$= \frac{e^{(\alpha+(2n+1)\beta)i} - e^{(\alpha-\beta)i}}{2i \sin \beta}$$

compare the real and imaginary parts on both sides:

$$\sum_{\nu=0}^{n} \cos (\alpha + 2\nu\beta) = \frac{\sin (\alpha + (2n + 1)\beta) - \sin (\alpha - \beta)}{2 \sin \beta},$$
$$\sum_{\nu=0}^{n} \sin (\alpha + 2\nu\beta) = \frac{\cos (\alpha + \beta) - \cos (\alpha + (2n + 1)\beta)}{2 \sin \beta}.$$

From the first of these, we obtain the formula in the text by making $\alpha = 0$, $\beta = \pi(t - x)$.

In the special case $f(x) = 1$, (34) gives $a_0 = 2$, $a_\nu = b_\nu = 0$ $(\nu > 0)$, and (36) becomes

$$1 = \int_0^1 \frac{\sin (2n+1)\pi(t-x)}{\sin \pi(t-x)} dt;$$

multiplying this by $f(x)$ and subtracting from (36), it is seen that

$$\begin{aligned} &\frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos 2\nu\pi x + b_\nu \sin 2\nu\pi x) - f(x) \\ (37) \qquad &= \int_0^1 (f(t) - f(x)) \frac{\sin (2n+1)\pi(t-x)}{\sin \pi(t-x)} dt \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where J_1, J_2 and J_3 are the integrals corresponding to the decomposition of the interval $(0, 1)$ into the three intervals $(0, \delta)$, $(\delta, 1-\delta)$ and $(1-\delta, 1)$. Our theorem will then be proved if we show that after assigning an arbitrarily small ϵ, a δ may be chosen sufficiently small and an n_0 sufficiently large, both independent of x, such that $|J_1| + |J_2| + |J_3|$ will be as small as we please for $n > n_0$ and every x in $(\epsilon, 1-\epsilon)$. Assume $0 < \delta < \epsilon/2$; for $\epsilon \leqq x \leqq 1-\epsilon$ and $0 \leqq t \leqq \delta$ or $1-\delta \leqq t \leqq 1$, we then have $\epsilon/2 < |t-x| < 1-(\epsilon/2)$ and consequently $|\sin \pi(t-x)| > \sin (\pi\epsilon/2)$; since $|f(t) - f(x)| \leqq |f(t)| + |f(x)|$, the expression under the integral sign in J_1 and J_3 will be less than $(|f(t)| + |f(x)|)/\sin \pi(\epsilon/2)$, whence

$$|J_1| + |J_3| < \frac{1}{\sin(\pi\epsilon/2)}\left[\int_0^\delta (|f(t)| + |f(x)|)dt + \int_{1-\delta}^1 (|f(t)| + |f(x)|)dt\right].$$

As $f(x)$ is holomorphic in $(\epsilon, 1-\epsilon)$, there exists a constant M such that $|f(x)| < M$ for all x in $(\epsilon, 1-\epsilon)$, whence

$$|J_1| + |J_3| < \frac{1}{\sin(\pi\epsilon/2)}\left(2\delta M + \int_0^\delta |f(t)|\,dt + \int_{1-\delta}^1 |f(t)|\,dt\right),$$

Since $\int_0^1 |f(t)|\,dt$ exists by hypothesis, the last two integrals may be made as small as we please by taking δ sufficiently small, and the same being true of $2\delta M$, it follows that $|J_1| + |J_3|$ may be made as small as we please by taking δ sufficiently small and independent of x. In J_2, the function

$$\frac{f(t) - f(x)}{\sin \pi(t-x)} = \varphi(x, t)$$

is evidently holomorphic for $\epsilon \leqq x \leqq 1-\epsilon$, $\delta \leqq t \leqq 1-\delta$, so that con-

stants M_1 and M_2 exist such that $|\varphi(x, t)| < M_1$ and

$$\left|\frac{\partial\varphi(x, t)}{\partial t}\right| < M_2$$

for all these values of x and t. Integrating by parts,

$$J_2 = \int_\delta^{1-\delta} \varphi(x, t) \sin(2n+1)\pi(t-x)dt$$
$$= -\left[\frac{1}{(2n+1)\pi}\varphi(x, t)\cos(2n+1)\pi(t-x)\right]_{t=\delta}^{t=1-\delta}$$
$$+ \frac{1}{(2n+1)\pi}\int_\delta^{1-\delta}\frac{\partial\varphi(x, t)}{\partial t}\cos(2n+1)\pi(t-x)dt,$$

and since $|\varphi(x, t)\cos(2n+1)\pi(t-x)| \leqq |\varphi(x, t)| < M_1$,

$$\left|\frac{\partial\varphi(x, t)}{\partial t}\cos(2n+1)\pi(t-x)\right| < M_2,$$

we have

$$|J_2| < \frac{2M_1 + M_2}{(2n+1)\pi}.$$

Consequently, after fixing a δ so small that $|J_1| + |J_2| < \eta$, where η is as small as we please, M_1 and M_2, which depend on δ but not on x, are fixed, and the last inequality now permits us to determine an n_0 (dependent on δ but independent of x) such that $|J_2| < \eta$ for $n \geqq n_0$ and all x in $(\epsilon, 1-\epsilon)$. The expression to the left in (37) is then less than 2η in absolute value, which proves our theorem.

We shall now derive a formula which is useful in estimating the remainder in some series occurring in the next chapters: Let

$$c_n \geqq c_{n+1} \geqq \cdots \geqq c_{n+p-1} \geqq c_{n+p} \geqq 0,$$

k a constant and $\epsilon > 0$ but as small as we please, then

$$(38) \qquad \left|\sum_{\nu=n}^{n+p} c_\nu \sin(2\nu+k)\pi x\right| \leqq \frac{c_n}{\sin\pi\epsilon}, \qquad \epsilon \leqq x \leqq 1-\epsilon.$$

Using the identity

$$2\sin\pi x\sin(2\nu+k)\pi x = \cos(2\nu-1+k)\pi x - \cos(2\nu+1+k)\pi x,$$

we have

$$2\sin\pi x\sum_{\nu=n}^{n+p} c_\nu\sin(2\nu+k)\pi x$$
$$= \sum_{\nu=n}^{n+p} c_\nu\cos(2\nu-1+k)\pi x - \sum_{\nu=n}^{n+p} c_\nu\cos(2\nu+1+k)\pi x$$

$$= c_n \cos(2n - 1 + k)\pi x - \sum_{\nu=n+1}^{n+p} (c_{\nu-1} - c_\nu) \cos(2\nu - 1 + k)\pi x$$
$$- c_{n+p} \cos(2n + 2p + 1 + k)\pi x,$$

and taking the sum of the absolute values of all terms to the right and remembering that $c_{\nu-1} - c_\nu \geqq 0$,

$$\left| 2 \sin \pi x \sum_{\nu=n+1}^{n+p} c_\nu \sin(2\nu + k)\pi x \right| \leqq c_n + \sum_{\nu=n+1}^{n+p} (c_{\nu-1} - c_\nu) + c_{n+p} = 2c_n,$$

whence (38) follows at once.

8. **Bernoulli functions and numbers.** We now apply the theorem of the preceding paragraph to $f(x) = \frac{1}{2} - x$; (34) gives, upon integration by parts, $a_0 = 0$, $a_n = 0$, $b_n = 1/n\pi$ $(n \geqq 1)$, whence

$$\tfrac{1}{2} - x = \sum_{1}^{\infty} \frac{\sin 2n\pi x}{n\pi}, \qquad 0 < x < 1,$$

the series being uniformly convergent for $\epsilon \leqq x \leqq 1 - \epsilon$. The series to the right converges also for $x = 0$ and $x = 1$, the sum being zero in either case, while the left side becomes $\pm \frac{1}{2}$. Let $[x]$ be the greatest integer contained in x, so that $0 \leqq x - [x] < 1$, then $\frac{1}{2} + [x] - x$ remains unchanged when an integer is added to x, and equals $\frac{1}{2} - x$ for $0 < x < 1$. Since the Fourier series to the right also has the period unity, we see that if we write

$$P_1(x) = \tfrac{1}{2} + [x] - x, \tag{39}$$

then for any x which is not an integer

$$P_1(x) = \sum_{1}^{\infty} \frac{\sin 2n\pi x}{n\pi}, \tag{40}$$

while for an integer m, $P_1(m + \delta) \to \frac{1}{2}$ and $P_1(m - \delta) \to -\frac{1}{2}$ as $\delta \to 0$ through positive values. Now write

$$P_2(x) = \sum_{1}^{\infty} \frac{\cos 2n\pi x}{2n^2\pi^2},$$

the series being uniformly convergent for all values of x, since its general term is less than $1/n^2$ in absolute value. Then, since (40) converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, we have

$$\frac{dP_2(x)}{dx} = -P_1(x) = x - \tfrac{1}{2}$$

(theorem XXI cor.) and therefore

$$P_2(x) = \frac{x^2 - x}{2} + c$$

in the latter interval; to determine the constant of integration, we observe that, on account of the uniform convergence of the series,

$$\int_0^1 P_2(x)dx = \int_0^1 \sum_1^\infty \frac{\cos 2n\pi x}{2n^2\pi^2}dx = \sum_1^\infty \int_0^1 \frac{\cos 2n\pi x}{2n^2\pi^2}dx = 0,$$

and therefore

$$\int_0^1 \left(\frac{x^2 - x}{2} + c\right)dx = 0 \qquad \text{or} \qquad c = \tfrac{1}{12}.$$

Now the polynomial $\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ takes the same value $(= \frac{1}{12})$ for $x = 0$ and $x = 1$; consequently the expression $\frac{1}{2}(x - [x])^2 - \frac{1}{2}(x - [x]) + \frac{1}{12}$ admits the period unity and is continuous for all values of x, so that the relation

$$P_2(x) = \tfrac{1}{2}(x - [x])^2 - \tfrac{1}{2}(x - [x]) + \tfrac{1}{12} = \sum_1^\infty \frac{\cos 2n\pi x}{2n^2\pi^2},$$

which we proved for $0 < x < 1$, subsists for *all* values of x since both members are continuous everywhere and admit the period unity. We observe that the integral values of x form no exception as in the case of $P_1(x)$.

Continuing this process, we define functions $P_k(x)$, $k = 2, 3, \cdots$, by the relations

$$P_k(x) = (-1)^k \frac{dP_{k+1}(x)}{dx},$$

$$\int_0^1 P_{k+1}(x)dx = 0, \tag{41}$$

$$P_{k+1}(x + 1) = P_{k+1}(x),$$

and show by complete induction that $P_k(x)$ is a polynomial of degree k in $x - [x]$ with rational coefficients, that $P_k(x)$ is continuous for all values of x when $k > 1$, since $P_{k+1}(1) - P_{k+1}(0) = (-1)^k \int_0^1 P_k(x)dx = 0$, and that

$$P_{2k}(x) = \sum_1^\infty \frac{\cos 2n\pi x}{2^{2k-1}n^{2k}\pi^{2k}},$$

$$P_{2k+1}(x) = \sum_1^\infty \frac{\sin 2n\pi x}{2^{2k}n^{2k+1}\pi^{2k+1}}, \tag{42}$$

these series being absolutely and uniformly convergent for all values of x.

From (42) it follows that, when m is an integer

$$P_{2k+1}(m) = P_{2k+1}(m + \tfrac{1}{2}) = 0,$$

$$(43) \qquad P_{2k}(m) = \frac{1}{2^{2k-1}\pi^{2k}} \sum_1^\infty \frac{1}{n^{2k}},$$

$$P_{2k}(m + \tfrac{1}{2}) = -\left(1 - \frac{1}{2^{2k-1}}\right) P_{2k}(m),$$

the last relation resulting from the identity

$$\sum_1^\infty \frac{(-1)^n}{n^{2k}} = 2\sum_1^\infty \frac{1}{(2n)^{2k}} - \sum_1^\infty \frac{1}{n^{2k}} = -\left(1 - \frac{1}{2^{2k-1}}\right)\sum_1^\infty \frac{1}{n^{2k}}.$$

Furthermore (42) gives

$$(44) \qquad P_k(1 - x) = (-1)^k P_k(x) = P_k(-x).$$

We shall now show that *for k odd and > 1, the equation $P_k(x) = 0$ has the roots $0, \frac{1}{2}, 1$ and that $P_k(x) > 0$ for $0 < x < \frac{1}{2}$, $P_k(x) < 0$ for $\frac{1}{2} < x < 1$, while for k even, there is one and only one root interior to each of the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.* From (43) it is seen that $P_k(0) = P_k(\frac{1}{2}) = P_k(1) = 0$ for k odd, but that for k even none of these expressions vanishes. For $k = 2$, our theorem is true, since $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 12$ in $(0, 1)$; we may therefore assume the theorem proved up to the index $k - 1$ and deduce its validity for the index k in the following manner. If, for k odd, $P_k(x) = 0$ has a root in $(0, 1)$ distinct from $0, \frac{1}{2}, 1$, we may assume, by (44), that it is interior to $(0, \frac{1}{2})$. But since

$$P_{k-1}(x) = \frac{dP_k(x)}{dx}$$

by (41), $P_{k-1}(x) = 0$ would then, according to Rolle's theorem, have two roots interior to the interval $(0, \frac{1}{2})$, contrary to the proven facts for $k - 1$ even. Thus $P_k(x)$ does not change its sign for $0 < x < \frac{1}{2}$, and since for $x = 0$,

$$\frac{dP_k(x)}{dx} = P_{k-1}(0)$$

is positive by (43), we find that $P_k(x) > 0$ for $0 < x < \frac{1}{2}$, and (44) then gives $P_k(x) < 0$ for $\frac{1}{2} < x < 1$. If, for k even, $P_k(x) = 0$ has two roots interior to $(0, \frac{1}{2})$, $P_{k-1}(x) = 0$ would have a root interior to the same interval, contrary to the proven facts for $k - 1$ odd. The same argument applies to the interval $(\frac{1}{2}, 1)$.

The expressions $P_{2k}(m) = P_{2k}(0)$ of (43) also occur in the expansion of the cotangent in a power series. From the decomposition of the co-

tangent into partial fractions we obtain

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{n=1}^{\infty}\frac{2t^2}{4\pi^2n^2 - t^2},$$

the series being uniformly convergent for $|t| \leqq 2\pi - \epsilon$. For such values of t, we have

$$\frac{2t^2}{4\pi^2n^2 - t^2} = \sum_{=1}^{\infty}\frac{t^{2k}}{2^{2k-1}\pi^{2k}n^{2k}},$$

and by Weierstrass's theorem on series of analytic functions, we obtain

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{k=1}^{\infty} t^{2k}\cdot\sum_{n=1}^{\infty}\frac{1}{2^{2k-1}\pi^{2k}n^{2k}}.$$

We now define the kth Bernoulli number B_k by[4]

$$B_k = \frac{(2k)!}{2^{2k-1}\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}, \tag{45}$$

and obtain[5]

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{k=1}^{\infty}\frac{B_k t^{2k}}{(2k)!}. \tag{46}$$

In

$$\frac{t}{2}\cot\frac{t}{2} = \frac{ti}{2}\,\frac{e^{ti/2} + e^{-ti/2}}{e^{ti/2} - e^{-ti/2}} = \frac{ti}{2} + \frac{ti}{e^{ti} - 1}$$

we now replace t by $-ti$; (46) then becomes

$$1 - \frac{t}{e^t - 1} = \frac{t}{2} + \sum_{k=1}^{\infty}\frac{(-1)^k B_k t^{2k}}{(2k)!}, \qquad |t| < 2\pi. \tag{47}$$

Multiplying both sides by

$$e^t - 1 = \sum_{1}^{\infty}\frac{t^k}{k!},$$

we find

$$\sum_{1}^{\infty}\frac{t^k}{k!} - t = \left(\sum_{1}^{\infty}\frac{t^k}{k!}\right)\left(\frac{t}{2} + \sum_{1}^{\infty}\frac{(-1)^k B_k t^{2k}}{(2k)!}\right)$$

and comparing coefficients of t^{2k+1} on both sides

$$\frac{1}{(2k+1)!} = \frac{1}{2(2k)!} - \frac{B_1}{(2k-1)!2!} + \frac{B_2}{(2k-3)!4!}$$
$$- \frac{B_3}{(2k-5)!6!} + \cdots + \frac{(-1)^k B_k}{1!(2k)!};$$

[4] These numbers appear for the first time in Jacob Bernoulli's Ars conjectandi, Basel, 1713, p. 97, in connection with the problem of expressing the sum $1^k + 2^k + 3^k + \cdots + (n-1)^k$ as a polynomial in n.

[5] Euler, L., Introductio in analysis infinitorum, Lausanne, 1748, p. 161.

this recurrent formula allows us to calculate successively B_1, B_2, $\cdots$ which are clearly all rational numbers. The first ten Bernoulli numbers are thus found to be

$$B_1 = \tfrac{1}{6},\quad B_2 = \tfrac{1}{30},\quad B_3 = \tfrac{1}{42},\quad B_4 = \tfrac{1}{30},\quad B_5 = \tfrac{5}{66},\quad B_6 = \tfrac{691}{2730},$$
$$B_7 = \tfrac{7}{6},\quad B_8 = \tfrac{3617}{510},\quad B_9 = \tfrac{43867}{798},\quad B_{10} = \tfrac{174611}{330}.$$

The comparison of (43) and (45) gives, since $P_k(m) = P_k(0)$ for $k > 1$,

$$P_{2k+1}(0) = P_{2k+1}(\tfrac{1}{2}) = 0,$$
$$P_{2k}(0) = \frac{1}{(2k)!}B_k, \qquad P_{2k}(\tfrac{1}{2}) = -\frac{1}{(2k)!}\left(1 - \frac{1}{2^{2k-1}}\right)B_k. \tag{48}$$

The left side of (47) is the special case $x = 0$ of $1 - [(te^{tx})/(e^t - 1)]$, and the latter expression may evidently be expanded in a power series in t, the coefficients being polynomials in x, the series converging uniformly for $|t| \leqq 2\pi - \epsilon$ and $|x| < a$, where ϵ is as small and a as large as we please. We write this expansion

$$1 - \frac{te^{tx}}{e^t - 1} = \sum_{k=1}^{\infty}(-1)^{k(k-1)/2}\varphi_k(x)t^k. \tag{49}$$

Since

$$1 - \frac{te^{tx}}{e^t - 1} = 1 + e^{tx}\left(1 - \frac{t}{e^t - 1} - 1\right),$$

we obtain from (47) and (49)

$$\sum_1^{\infty}(-1)^{k(k-1)/2}\varphi_k(x)t^k = 1 + \left(\sum_0^{\infty}\frac{x^k t^k}{k!}\right)\left(-1 + \frac{t}{2} + \sum_1^{\infty}(-1)^k\frac{B_k t^{2k}}{(2k)!}\right),$$

and upon multiplication and comparison of coefficients of like powers of t,

$$\begin{aligned}(-1)^k\varphi_{2k}(x) = & -\frac{x^{2k}}{(2k)!} + \frac{x^{2k-1}}{2(2k-1)!} - \frac{B_1x^{2k-2}}{(2k-2)!2!} + \frac{B_2x^{2k-4}}{(2k-4)!4!} \\ & - \frac{B_3x^{2k-6}}{(2k-6)!6!} + \cdots + (-1)^{k-1}\frac{B_{k-1}x^2}{2!(2k-2)!} \\ & + (-1)^k\frac{B_k}{(2k)!},\end{aligned}$$

$$\begin{aligned}(-1)^k\varphi_{2k+1}(x) = & -\frac{x^{2k+1}}{(2k+1)!} + \frac{x^{2k}}{2(2k)!} - \frac{B_1x^{2k-1}}{(2k-1)!2!} + \frac{B_2x^{2k-3}}{(2k-3)!4!} \\ & - \frac{B_3x^{2k-5}}{(2k-5)!6!} + \cdots + (-1)^{k-1}\frac{B_{k-1}x^3}{3!(2k-2)!} \\ & + (-1)^k\frac{B_kx}{(2k)!}\end{aligned}$$

for $k > 0$, while $\varphi_1(x) = \frac{1}{2} - x$, so that for $0 < x < 1$, $\varphi_1(x) = P_1(x)$. We shall now prove that for $k > 1$ and $0 \leqq x \leqq 1$

$$\varphi_k(x) = P_k(x).$$

Since for $0 \leqq x \leqq 1$, the $P_k(x)$ are completely determined by the first two relations (41), it is evidently sufficient to show that the $\varphi_k(x)$ satisfy the same relations. Differentiating (49) in respect to x, we obtain, since the differentiation of the right hand series term by term is legitimate according to Weierstrass's theorem on series of analytic functions,

$$\sum_1^\infty (-1)^{k(k-1)/2} \frac{d\varphi_k(x)}{dx} t^k = -\frac{t^2 e^{tx}}{e^t - 1} = t\left(\sum_1^\infty (-1)^{k(k-1)/2} \varphi_k(x) t^k - 1\right),$$

and comparing coefficients of t^{k+1}

$$(-1)^k \frac{d\varphi_{k+1}(x)}{dx} = \varphi_k(x).$$

Integrating (49) from $x = 0$ to $x = 1$, we find

$$\sum_1^\infty (-1)^{k(k-1)/2} t^k \int_0^1 \varphi_k(x)dx = \left[x - \frac{e^{tx}}{e^t - 1}\right]_{x=0}^{x=1} = 0,$$

whence

$$\int_0^1 \varphi_k(x)dx = 0.$$

Outside of (0, 1), $\varphi_k(x)$ and $P_k(x)$ do not coincide, since $P_k(x)$ has the period unity, but $\varphi_k(x)$ has not. To find the difference $\varphi_k(x+1) - \varphi_k(x)$, change x into $x + 1$ in (49) and subtract:

$$\sum_1^\infty (-1)^{k(k-1)/2} (\varphi_k(x+1) - \varphi_k(x)) t^k = -te^{tx} = -\sum_1^\infty \frac{x^{k-1} t^k}{(k-1)!},$$

whence

$$\varphi_k(x+1) - \varphi_k(x) = -(-1)^{k(k-1)/2} \frac{x^{k-1}}{(k-1)!}.$$

Changing k into $k + 1$ and adding these equations for $x = 0, 1, 2, \cdots, n - 1$, we find

$$1^k + 2^k + 3^k + \cdots + (n-1)^k = (-1)^{k(k-1)/2} k!(\varphi_{k+1}(n) - \varphi_{k+1}(0)),$$

which is essentially[6] the formula established by Bernoulli.

[6] The Bernoulli polynomials (as distinct from the periodic Bernoulli *functions* $P_k(x)$) are commonly defined by

$$t \cdot \frac{e^{tx} - 1}{e^t - 1} = \sum_1^\infty \bar{\varphi}_k(x) \frac{t^k}{k!},$$

and by comparison to (49), we readily find the relation $\bar{\varphi}_k(x) = (-1)^{k(k-1)/2}(\varphi_k(0) - \varphi_k(x))$ between this definition and that of the text; the latter has been chosen in view of its application to Euler's summation formula in the next paragraph. For further information on Bernoulli numbers and polynomials, see Saalschütz, L., Vorlesungen über die Bernoullischen Zahlen, Berlin, 1893.

9. Euler's summation formula. Let $F(s)$ be a real or complex function of the real variable s, having continuous derivatives up to the order $2m$ inclusive, or we may suppose s complex and $F(s)$ an analytic function; furthermore let $0 \leqq a < 1$ and $b > 0$. In the identity

$$F(s + nb) - F(s + ab + \nu b) = b \int_{a+\nu}^{n} F'(s + tb)dt$$

make $\nu = 0, 1, 2, \cdots, n - 1$ and add, obtaining

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b) = b \sum_{\nu=0}^{n-1} \int_{\nu+a}^{n} F'(s + tb)dt.$$

Decomposing each integral by the formula

$$\int_{\nu+a}^{n} F'(s + tb)dt = \int_{\nu+a}^{\nu+1} F'(s + tb)dt + \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt$$

we obtain

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b)$$
$$= b \sum_{\nu=0}^{n-1} \int_{\nu+a}^{\nu+1} F'(s + tb)dt + b \sum_{\nu=0}^{n-1} \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt;$$

in the double sum, each integral between limits μ and $\mu + 1$ occurs μ times, namely for $\nu = 0, 1, 2, \cdots, \mu - 1$, so that

$$\sum_{\nu=0}^{n-1} \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt = \sum_{\mu=1}^{n-1} \mu \int_{\mu}^{\mu+1} F'(s + tb)dt,$$

or adding to the last sum the zero term corresponding to $\mu = 0$, and writing ν instead of μ

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b)$$
$$= b \sum_{\nu=0}^{n-1} \left(\int_{\nu+a}^{\nu+1} F'(s + tb)dt + \nu \int_{\nu}^{\nu+1} F'(s + tb)dt \right).$$

In the integrals with lower limit $\nu + a$, we may reduce this limit to ν by observing that for $\nu \leqq t < \nu + a$, $[t - a] = \nu - 1$, or $[t - a] - \nu + 1 = 0$, while for $\nu + a \leqq t < \nu + 1$, $[t - a] = \nu$, or $[t - a] - \nu + 1 = 1$; consequently

$$\int_{\nu}^{\nu+1} ([t - a] - \nu + 1)F'(s + tb)dt = \int_{\nu+a}^{\nu+1} F'(s + tb)dt,$$

and substituting in the previous equation

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b) = b \sum_{\nu=0}^{n-1} \int_{\nu}^{\nu+1} ([t - a] + 1)F'(s + tb)dt$$
$$= b \int_{0}^{n} ([t - a] + 1)F'(s + tb)dt.$$

But from (39), $[t - a] + 1 = t - a + \frac{1}{2} + P_1(t - a)$, and integrating by parts, we find

$$b\int_0^n (t - a + \tfrac{1}{2})F'(s + tb)dt$$

$$= (n - a + \tfrac{1}{2})F(s + nb) - (\tfrac{1}{2} - a)F(s) - \int_0^n F(s + tb)dt;$$

introducing all this in the preceding formula, we obtain

$$\sum_{\nu=0}^{n-1} F(s + ab + \nu b) = \int_0^n F(s + tb)dt + (\tfrac{1}{2} - a)(F(s) - F(s + nb)) - b\int_0^n P_1(t - a)F'(s + tb)dt. \tag{50}$$

Integrating the last integral by parts, using the first and third of (41) and the second part of (44), we see at once by complete induction that

$$\begin{aligned}\sum_{\nu=0}^{n-1} F(s + ab + \nu b) &= \int_0^n F(s + tb)dt + (\tfrac{1}{2} - a)(F(s) - F(s + nb)) \\ &+ \sum_{k=1}^{2m-1} (-1)^{[k(k+1)/2]+1} b^k P_{k+1}(a)(F^{(k)}(s + nb) - F^{(k)}(s)) \\ &+ (-1)^m b^{2m}\int_0^n P_{2m}(t - a)F^{(2m)}(s + tb)dt,\end{aligned} \tag{51}$$

$$0 \leqq a < 1, \qquad b > 0.$$

In the special case $a = 0$, the values of $P_{k+1}(a)$ are given by (48), and adding $F(s + nb)$ to both sides in (51), we obtain Euler's summation formula[7]

$$\begin{aligned}\sum_{\nu=0}^{n} F(s + \nu b) &= \int_0^n F(s + tb)dt + \tfrac{1}{2}(F(s + nb) + F(s)) \\ &+ \sum_{k=1}^{m} \frac{(-1)^{k-1}B_k}{(2k)!} b^{2k-1}(F^{(2k-1)}(s + nb) - F^{(2k-1)}(s)) \\ &+ (-1)^m b^{2m}\int_0^n P_{2m}(t)F^{(2m)}(s + tb)dt.\end{aligned} \tag{52}$$

[7] Euler, L., Methodus generalis summandi progressiones. Comment. Acad. Petrop., vol. 6 (1732–33, published 1738), pp. 68–97, and independently, Maclaurin, C., A treatise on fluxions. Edinburgh, 1742, art. 352 and 828–30, 847. Neither author gives the remainder term. The proof in the text was given, in the special case $a = 0$, by Wirtinger, W., Einige Anwendungen der Euler-Maclaurinschen Summenformel, insbesondere auf eine Aufgabe von Abel. Acta Math., vol. 26 (1902), pp. 255–271.

In view of the application of (51) to the approximate evaluation of the sum to the left, it will be necessary to investigate the remainder term, or the last term to the right. The term corresponding to $k = 2m - 1$ in the sum to the right in (51) is

$$(-1)^{m-1}b^{2m-1}P_{2m}(a)(F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)) = (-1)^{m-1}b^{2m}P_{2m}(a)\int_0^n F^{(2m)}(s+tb)dt,$$

so that (51) may be written in the form

$$\begin{aligned}\sum_{\nu=0}^{n-1} F(s+ab+\nu b) &= \int_0^n F(s+tb)dt + (\tfrac{1}{2} - a)(F(s) - F(s+nb)) \\ &+ \sum_{k=1}^{2m-2}(-1)^{[k(k+1)/2]+1}b^k P_{k+1}(a)(F^{(k)}(s+nb) - F^{(k)}(s)) + R_{2m},\end{aligned} \tag{53}$$

$$R_{2m} = (-1)^m b^{2m}\int_0^n (P_{2m}(t-a) - P_{2m}(a))F^{(2m)}(s+tb)dt. \tag{54}$$

Now suppose s and $F(s)$ *real*, and assume that *the sign of* $F^{(2m)}(s+tb)$ *does not change for* $0 \leqq t \leqq n$, so that, writing $\alpha = 1$ or -1, $\alpha F^{2m}(s+tb) \geqq 0$ in the interval of integration in (54). Since $P_{2m}(t)$ takes its maximum value for $t = 0$, we have

$$|P_{2m}(t-a) - P_{2m}(a)| \leqq |P_{2m}(t-a)| + |P_{2m}(a)| < 2P_{2m}(0),$$

and by (14) and (13),

$$\begin{aligned}\left|\int_0^n (P_{2m}(t-a) - P_{2m}(a))\cdot\alpha F^{(2m)}(s+tb)dt\right| &\leqq \int_0^n 2P_{2m}(0)\cdot\alpha F^{(2m)}(s+tb)dt \\ &= \frac{2P_{2m}(0)}{b}\alpha(F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)),\end{aligned}$$

so that[8]

$$|R_{2m}| < 2b^{2m-1}P_{2m}(0)\,|F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)|. \tag{55}$$

In the two most important special cases $a = 0$ and $a = \frac{1}{2}$, we shall now obtain more accurate approximate values of R_{2m} by introducing stronger assumptions regarding $F^{(2m)}(s)$.[9] Let $\alpha = \pm 1$, and assume *first, that*

[8] For $a = 0$, a slightly more accurate formula was given by Poisson, S. D., "Sur le calcul numérique des Intégrales définies," Mém. Ac. Sc. Paris, vol. 6 (1827), pp. 571–602. Other investigations of the remainder term (in the special case $a = 0$) by Jacobi, C. G. J., "De uso legitimo formulæ summatoriæ Maclaurinianæ," Journ. f. Math., vol. 12 (1834), pp. 263–272, and Malmsten, C. J., "Sur la formule, etc.," Acta Math., vol. 5 (1884), pp. 1–46 (corrected reprint from Journ. f. Math., vol. 35 (1847), pp. 55–82).

[9] For $a = 0$, this method is due to Sonin, N., "Sur les termes complémentaires de la formule sommatoire d'Euler et de celle de Stirling," Ann. de l'Ec. Norm., ser. 3, vol. 6 (1889), pp. 257–262.

$\alpha F^{(2m)}(s + tb)$ is positive and decreasing as t increases from 0 *to $n + \frac{1}{2}$, and second, that*

$$\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb))$$

is positive and decreasing as t increases from 0 to $\frac{1}{2}$. *Then, for $a = 0$,*

$$(-1)^{m-1}R_{2m} = b^{2m-1}P_{2m}(0)(F^{(2m-1)}(s + nb + hb) - F^{(2m-1)}(s + hb)), \quad 0 < h < \tfrac{1}{2}. \tag{56}$$

Since $a = 0$, we may write (54) in the form

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^n (P_{2m}(0) - P_{2m}(t))\alpha F^{(2m)}(s + tb)dt,$$

and since $P_{2m}(t)$ has the period 1,

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^1 (P_{2m}(0) - P_{2m}(t))\alpha \sum_{\nu=0}^{n-1} F^{(2m)}(s + \nu b + tb)dt.$$

Decomposing the integral in two with limits 0, $\frac{1}{2}$ and $\frac{1}{2}$, 1 respectively, and introducing $1 - t$ as integration variable in the latter, we find by the aid of (44) that

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^{\frac{1}{2}} (P_{2m}(0) - P_{2m}(t))\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb)dt. \tag{57}$$

To obtain an upper bound for this expression, we shall use a theorem due to Tchebychef: *When $u(t)$ and $v(t)$ are continuous for $a \leqq t \leqq b$, and both these functions increase (or both decrease) as t increases, then*

$$(b - a)\int_a^b u(t)v(t)dt > \int_a^b u(t)dt \cdot \int_a^b v(t)dt,$$

but

$$(b - a)\int_a^b u(t)v(t)dt < \int_a^b u(t)dt \cdot \int_a^b v(t)dt$$

when one of the functions increases and the other decreases.[10] Writing

[10] The following simple proof is due to Franklin, F., "Proof of a Theorem of Tchebycheff's on Definite Integrals," Am. Journ., vol. 7, pp. 377–379: Consider the double integral

$$\int_a^b\int_a^b (u(t) - u(x))(v(t) - v(x))dtdx = \int_a^b dt \int_a^b [u(t)v(t) - u(t)v(x) - u(x)v(t) + u(x)v(x)]dx$$
$$= (b - a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt \cdot \int_a^b v(x)dx - \int_a^b v(t)dt \cdot \int_a^b u(x)dx + (b - a)\int_a^b u(x)v(x)dx,$$

or replacing x by t in the single integrals to the right

$$\frac{1}{2}\int_a^b\int_a^b (u(t) - u(x))(v(t) - v(x))dtdx = (b - a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt \cdot \int_a^b v(t)dt.$$

When $u(t)$ and $v(t)$ vary in the same sense, $u(t) - u(x)$ and $v(t) - v(x)$ have the same sign, so

$u(t) = P_{2m}(0) - P_{2m}(t)$, we have, by (41), $u'(t) = P_{2m-1}(t) > 0$ for $0 < t < \frac{1}{2}$, so that $u(t)$ is positive and increases with t in $(0, \frac{1}{2})$, and by (41) and (43),

$$\int_0^{\frac{1}{2}} u(t)dt = [P_{2m}(0)\cdot t - P_{2m+1}(t)]_{t=0}^{t=\frac{1}{2}} = \tfrac{1}{2}P_{2m}(0).$$

The second factor in (57),

$$v(t) = \alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb)),$$

is positive and decreasing by hypothesis, and

$$\int_0^{\frac{1}{2}} v(t)dt = \int_0^n \alpha F^{(2m)}(s + tb)dt;$$

consequently, by Tchebychef's theorem,

$$P_{2m}(0)\int_0^n \alpha F^{(2m)}(s + tb)dt > (-1)^{m-1}b^{-2m}\alpha R_{2m}. \tag{58}$$

To obtain a lower bound for (57), we observe that $u(t)$ is positive and $v(t)$ decreasing in the interval of integration, whence

$$\int_0^{\frac{1}{2}} u(t)v(t)dt > v(\tfrac{1}{2})\int_0^{\frac{1}{2}} u(t)dt = P_{2m}(0)\sum_{\nu=0}^{n-1} \alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b).$$

But $\alpha F^{(2m)}(s + \nu b + \frac{1}{2}b + tb)$ is positive and decreasing when t increases from 0 to 1, and consequently

$$\alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b) > \int_0^1 \alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b + tb)dt,$$

so that, by the preceding inequality,

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} > P_{2m}(0)\int_0^n \alpha F^{(2m)}(s + \tfrac{1}{2}b + tb)dt. \tag{59}$$

Since by the first hypothesis on $F^{(2m)}(s)$, the integral

$$\int_0^n \alpha F^{(2m)}(s + hb + tb)dt$$

is a continuous and decreasing function of h as h increases from 0 to $\frac{1}{2}$, the comparison of (58) and (59) shows that there exists an h interior to

that their product is positive, and therefore also the double integral, which proves the first part of the theorem. The second part follows from the first upon replacing $u(t)$ by $-u(t)$.

$(0, \frac{1}{2})$ such that

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = P_{2m}(0)\int_0^n \alpha F^{(2m)}(s + hb + tb)dt,$$

and evaluating the integral, we obtain (56).

Now let $\alpha = \pm 1$, and assume *first, that* $\alpha F^{(2m)}(s + tb)$ *is positive and decreasing as* t *increases from* 0 *to* $n + \frac{1}{2}$, *and second, that*

$$\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb) + F^{(2m)}(s + \tfrac{1}{2}b + \nu b - tb))$$

is positive and increasing as t *increases from* 0 *to* $\frac{1}{2}$. *Then, for* $a = \frac{1}{2}$,

$$(-1)^{m-1}R_{2m} = b^{2m-1}P_{2m}(\tfrac{1}{2})(F^{(2m-1)}(s + nb + hb) - F^{(2m-1)}(s + hb)), \tag{60}$$
$$0 < h < \tfrac{1}{2}.$$

We obtain from (54), in the case $a = \frac{1}{2}$,

$$(-1)^m b^{-2m}R_{2m} = \int_{-\frac{1}{2}}^{n-\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\cdot\alpha F^{(2m)}(s + \tfrac{1}{2}b + tb)dt$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\alpha \sum_{\nu=0}^{n-1} F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb)dt$$

and decomposing the integral in two with limits $-\frac{1}{2}$, 0 and 0, $\frac{1}{2}$ respectively, and introducing $-t$ as integration variable in the first, (44) gives

$$(-1)^m b^{-2m}R_{2m} = \int_0^{\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\cdot\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb)$$
$$+ F^{(2m)}(s + \tfrac{1}{2}b + \nu b - tb))dt.$$

Here the first factor under the integral sign is seen at once to be positive and decreasing, while the second factor is positive and increasing by hypothesis, and (60) is now readily proved in the same way as before.

CHAPTER II.

Definite integrals for the Gamma function derived from its definition as an infinite product.

10. **Expression of the Gamma function by Euler's integral of the second kind. Integrals for $P(s)$ and $Q(s)$.** Assume $\Re(s) > 0$ (when $s = x + yi$, $\Re(s) = x$ is the real part of s), let t be a real variable, and define t^s as $e^{s \log t}$, where $\log t$ denotes the real value of the logarithm of t; since

$$t^{s-1} = \frac{d}{dt}\left(\frac{t^s}{s}\right),$$

it follows that

$$\int_0^1 t^{s-1}dt = \lim_{\epsilon\to 0}\left[\frac{t^s}{s}\right]_{t=\epsilon}^{t=1} = \frac{1}{s}.$$

Replacing s by $s+1$ and subtracting the new integral from the previous one, we find

$$\int_0^1 t^{s-1}(1-t)dt = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)},$$

and, generally, from

$$\int_0^1 t^{s-1}(1-t)^{n-1}dt = \frac{(n-1)!}{s(s+1)\cdots(s+n-1)}$$

it follows that

$$\int_0^1 t^{s-1}(1-t)^n dt = \frac{(n-1)!}{s(s+1)\cdots(s+n-1)} - \frac{(n-1)!}{(s+1)(s+2)\cdots(s+n)}$$

$$= \frac{n!}{s(s+1)\cdots(s+n)},$$

or finally, replacing t by t/n,

$$\int_0^n t^{s-1}\left(1-\frac{t}{n}\right)^n dt = \frac{n!\,n^s}{s(s+1)\cdots(s+n)}.$$

For $n\to\infty$, the expression to the right approaches the limit $\Gamma(s)$, as is seen from ($J4''$) upon replacing n by $n+1$ and a by unity, and since $(1-t/n)^n \to e^{-t}$, the above equation suggests that

$$(61) \qquad \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0.$$[11]

To prove this, we must first establish the convergence of the infinite integral, which is known as Euler's integral of the second kind. Writing $s = x + yi$, and assuming $A \geqq x \geqq \epsilon > 0$, where A is as large and ϵ as small as we please, we have $|e^{-t}t^{s-1}| = e^{-t}t^{x-1}$, and for $0 < t \leqq 1$, $e^{-t}t^{x-1} < t^{\epsilon-1}$, while for t sufficiently large, $e^t > t^{A+1}$ and

$$e^{-t}t^{x-1} < t^{-(A+1)}t^{A-1} = t^{-2}.$$

Hence by theorem XVI corollary and theorem XIX, the integral in (61) exists uniformly for $A \geqq \Re(s) \geqq \epsilon > 0$. In the second place, it must be shown that the left and right members of (61) are equal, or that

$$\lim_{n\to\infty}\int_0^n t^{s-1}\left(1-\frac{t}{n}\right)^n dt = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0.$$

[11] For s real and positive, this expression is due to Euler (J^{10}).

We have

$$\int_0^\infty e^{-t}t^{s-1}dt - \int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n dt$$

$$= \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{s-1}dt + \int_n^\infty e^{-t}t^{s-1}dt,$$

and the second integral to the right approaches zero as $n \to \infty$, on account of the convergence of the integral in (61). Using the inequality [12]

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq 2\frac{t^2}{n}e^{-t} \qquad \text{for} \qquad 0 \leqq t \leqq n,$$

we find

$$\left| \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{s-1}dt \right| \leqq \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{x-1}dt$$

$$\leqq \int_0^n \frac{2t^2}{n} e^{-t}t^{x-1}dt < \frac{2}{n}\int_0^\infty e^{-t}t^{x+1}dt \to 0,$$

the last integral being convergent and independent of n. Thus the proof of (61) is complete.

Replacing t by at, where $a > 0$, we obtain from (61)

$$\text{(62)} \qquad \frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at}t^{s-1}dt, \qquad a > 0, \qquad \Re(s) > 0.$$

The integral

$$\int_0^\infty e^{-t}t^{s-1} \log t dt$$

converges uniformly for $A \geqq x \geqq \epsilon > 0$, since for t sufficiently small, $| e^{-t}t^{s-1} \log t | < t^{\epsilon-1} | \log t | < t^{(\epsilon/2)-1}$, and for t sufficiently large,

$$| e^{-t}t^{s-1} \log t | < t^{-2} \log t < t^{-3/2};$$

[12] Proof: for $b \geqq a \geqq 0$, we have

$$0 \leqq b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \leqq nb^{n-1}(b - a).$$

When $b = e^{-t/n}$, where $0 \leqq t \leqq n$, and $a = 1 - (t/n)$, we have

$$b - a = \frac{1}{2!}\left(\frac{t}{n}\right)^2 - \frac{1}{3!}\left(\frac{t}{n}\right)^3 + \cdots,$$

so that, the series being alternating with numerically decreasing terms,

$$0 \leqq b - a \leqq \frac{1}{2}\left(\frac{t}{n}\right)^2.$$

Consequently

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq ne^{-(1-1/n)t} \cdot \frac{1}{2}\left(\frac{t}{n}\right)^2 = \frac{e^{t/n}}{2} \cdot e^{-t}\frac{t^2}{n} \leqq \frac{e}{2}e^{-t}\frac{t^2}{n} \leqq 2e^{-t}\frac{t^2}{n}.$$

The inequality and its use in proving (61) are due to Watson, G. N., "An inequality associated with the Gamma function," Messenger of Math., vol. 45 (1915), pp. 28–30.

consequently by theorem XXI, we may differentiate (61) in respect to x under the integral sign, and since

$$\frac{\partial \Gamma(s)}{\partial x} = \frac{d\Gamma(s)}{ds},$$

we find

$$\frac{d\Gamma(s)}{ds} = \int_0^\infty e^{-t} t^{s-1} \log t dt, \qquad \Re(s) > 0, \tag{63}$$

and the repeated differentiation under the integral sign gives

$$\frac{d^n\Gamma(s)}{ds^n} = \int_0^\infty e^{-t} t^{s-1} (\log t)^n dt, \qquad \Re(s) > 0. \tag{64}$$

Integral expressions for the functions $P(s)$ and $Q(s)$ may be obtained in the following manner: the expansion

$$e^{-t} t^{s-1} = \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\nu!} t^{s-1+\nu}$$

converges uniformly for $0 < \epsilon \leqq t \leqq 1$, and therefore, by theorem XXI corollary

$$\int_\epsilon^1 e^{-t} t^{s-1} dt = \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\nu!} \frac{1 - \epsilon^{s+\nu}}{s+\nu}.$$

For $\Re(s) = x > 0$, the integral $\int_0^1 e^{-t} t^{s-1} dt$ converges, and since

$$|\epsilon^{s+\nu}| \leqq \epsilon^x, \qquad |s+\nu| = |x+\nu+yi| \geqq x+\nu \geqq x,$$

we have

$$\left| \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\nu!} \frac{\epsilon^{s+\nu}}{s+\nu} \right| \leqq \frac{\epsilon^x}{x} \sum_{\nu=0}^\infty \frac{1}{\nu!} \to 0 \text{ as } \epsilon \to 0,$$

so that, by the definition of $P(s)$ (J33),

$$P(s) = \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\nu!(s+\nu)} = \int_0^1 e^{-t} t^{s-1} dt, \qquad \Re(s) > 0. \tag{65}$$

Since $Q(s) = \Gamma(s) - P(s)$, (61) and (65) give

$$Q(s) = \int_1^\infty e^{-t} t^{s-1} dt. \tag{66}$$

As the integrand does not become infinite at the lower limit for any finite value of s, the last integral converges uniformly for $-A \leqq \Re(s) \leqq A$, where A is as large as we please, and consequently the entire function $Q(s)$ equals the integral (66) for all finite values of s, by theorem XXIII. The integrals (65) and (66) are due to Legendre and Prym (J[34, 32]).

11. The hypergeometric series $F(\alpha, \beta, \gamma; s)$ and its value for $s = 1$.

Euler's integral of the first kind. The name hypergeometric was applied by Wallis[13] to the power series in s

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}s + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}s^2 + \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}s^3 + \cdots,$$

where α, β and γ may have any real or complex values, except that γ must not equal zero or a negative integer (for when $\gamma = -n$, the $n+1$st and following coefficients become infinite). Using the notations

$$\begin{aligned} &(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \\ (67)\qquad &(\alpha, n) = (n-1)!n^\alpha/(\alpha)_n, \qquad (\alpha, n) \to \Gamma(\alpha) \text{ as } n \to \infty, \end{aligned}$$

already introduced in J § 14 and J § 2 respectively, we may write this series

$$(68)\qquad F(\alpha, \beta, \gamma;\ s) = \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu(\beta)_\nu}{(1)_\nu(\gamma)_\nu}s^\nu.$$

Many of the elementary functions are special cases of this series; for instance

$$(1+s)^n = F(-n, \beta, \beta;\ s), \qquad \log(1+s) = sF(1, 1, 2;\ -s),$$
$$e^s = \lim_{\beta\to\infty} F(1, \beta, 1;\ s/\beta);$$

the essential properties of the hypergeometric series, and its intimate connection with the Gamma function, were discovered by Gauss (J^6). By (67), the absolute value of the general term in (68) may be written

$$\left|\frac{(1, \nu)(\gamma, \nu)}{(\alpha, \nu)(\beta, \nu)}\cdot\frac{s^\nu}{\nu^{1+\gamma-\alpha-\beta}}\right| = \left|\frac{(1, \nu)(\gamma, \nu)}{(\alpha, \nu)(\beta, \nu)}\right|\cdot\frac{|s|^\nu}{\nu^{1+\Re(\gamma-\alpha-\beta)}},$$

and since the first factor approaches

$$\left|\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right|$$

as $\nu\to\infty$, and $\Sigma\, 1/\nu^{1+k}$ converges for $k > 0$, but diverges for $k \leqq 0$, it is evident that the series (68) converge absolutely for $|s| < 1$, but is divergent for $|s| > 1$, and that for $|s| = 1$, the series is absolutely convergent when $\Re(\gamma-\alpha-\beta) > 0$.[14] To find the value of $F(\alpha, \beta, \gamma;\ 1)$, which is a convergent series when $\Re(\gamma-\alpha-\beta) > 0$, we begin by

[13] Wallis, J., Arithmetica infinitorum, Oxford, 1656.

[14] By more elaborate methods, it may be shown that for $|s| = 1$, the series is (conditionally) convergent for $0 \geqq R(\gamma-\alpha-\beta) > -1$, divergent for $R(\gamma-\alpha-\beta) \leqq -1$, except for $s = 1$, where the series diverges for $R(\gamma-\alpha-\beta) \leqq 0$.

establishing the following relation between the three hypergeometric series $F(\alpha, \beta, \gamma; s)$, $F(\alpha, \beta, \gamma - 1; s)$ and $F(\alpha, \beta, \gamma + 1; s)$ for $|s| < 1$:

$$
\begin{aligned}
(69) \quad & \gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)s]F(\alpha, \beta, \gamma; s) \\
& \qquad + (\gamma - \alpha)(\gamma - \beta)sF(\alpha, \beta, \gamma + 1; s) \\
& \qquad - \gamma(\gamma - 1)(1 - s)F(\alpha, \beta, \gamma - 1; s) = 0.
\end{aligned}
$$

In fact, the constant term on the left equals $\gamma(\gamma - 1) - \gamma(\gamma - 1) = 0$, and the coefficient of s^n, where $n > 0$, is seen by (68) to be

$$
\begin{aligned}
& \gamma(\gamma - 1)\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n} - \gamma(2\gamma - \alpha - \beta - 1)\frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma)_{n-1}} \\
& \qquad + (\gamma - \alpha)(\gamma - \beta)\frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma + 1)_{n-1}} \\
& \qquad - \gamma(\gamma - 1)\left(\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma - 1)_n} - \frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma - 1)_{n-1}}\right)
\end{aligned}
$$

or denoting by

$$u_n = \frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma - 1)_n}$$

the coefficient of s^n in $F(\alpha, \beta, \gamma - 1; s)$,

$$
\begin{aligned}
& \frac{\gamma(\gamma - 1)u_n}{\gamma + n - 1}\left(\gamma - 1 + \frac{n(1 + \alpha + \beta - 2\gamma)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}\right. \\
& \qquad + \frac{n(\gamma - \alpha)(\gamma - \beta)}{(\alpha + n - 1)(\beta + n - 1)} - (\gamma + n - 1) \\
& \qquad \left. + \frac{n(\gamma + n - 2)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}\right) \\
& \qquad = \frac{n\gamma(\gamma - 1)u_n}{\gamma + n - 1}\left(-1 + \frac{(\gamma - \alpha)(\gamma - \beta)}{(\alpha + n - 1)(\beta + n - 1)}\right. \\
& \qquad \left. + \frac{(\alpha + \beta - \gamma + n - 1)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}\right) = 0.
\end{aligned}
$$

In (69) we now let s tend toward unity through positive and increasing values; since $F(\alpha, \beta, \gamma; 1)$ and $F(\alpha, \beta, \gamma + 1; 1)$ are convergent series on account of $\Re(\gamma - \alpha - \beta) > 0$, Abel's theorem on power series (J^{31}) shows that

$$\lim_{s \to 1} F(\alpha, \beta, \gamma; s) = F(\alpha, \beta, \gamma; 1)$$

and

$$\lim_{s \to 1} F(\alpha, \beta, \gamma + 1; s) = F(\alpha, \beta, \gamma + 1; 1).$$

Furthermore $(1 - s)F(\alpha, \beta, \gamma - 1; s) = 1 + \sum_{1}^{\infty}(u_\nu - u_{\nu-1})s^\nu$ for $|s| < 1$

and since

$$1 + \sum_1^{\infty} (u_\nu - u_{\nu-1}) = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(1, n)(\gamma - 1, n)}{(\alpha, n)(\beta, n)} \cdot \frac{1}{n^{\gamma-\alpha-\beta}} = 0$$

on account of $\Re(\gamma - \alpha - \beta) > 0$, the theorem in question gives

$$\lim_{s\to 1} (1 - s)F(\alpha, \beta, \gamma - 1; s) = 0.$$

Thus we finally obtain from (69)

$$F(\alpha, \beta, \gamma; 1) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1; 1)$$

and replacing γ by $\gamma + 1, \gamma + 2, \cdots, \gamma + n - 1$ and multiplying the corresponding equations

$$F(\alpha, \beta, \gamma; 1) = \frac{(\gamma - \alpha)_n(\gamma - \beta)_n}{(\gamma)_n(\gamma - \alpha - \beta)_n} F(\alpha, \beta, \gamma + n; 1)$$

$$= \frac{(\gamma, n)(\gamma - \alpha - \beta, n)}{(\gamma - \alpha, n)(\gamma - \beta, n)} F(\alpha, \beta, \gamma + n; 1).$$

The first factor to the right approaches the limit $\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)/\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)$ as $n \to \infty$, and for $n > |\gamma|$, we have

$$|F(\alpha, \beta, \gamma + n; 1) - 1| = \left| \sum_{\nu=1}^{\infty} \frac{(1)_\nu(\gamma + n)_\nu}{(\alpha)_\nu(\beta)_\nu} \right|$$

$$\leqq \sum_{\nu=1}^{\infty} \left| \frac{(\alpha)_\nu(\beta)_\nu}{(1)_\nu(\gamma + n)_\nu} \right| \leqq \sum_{\nu=1}^{\infty} \frac{(|\alpha|)_\nu(|\beta|)_\nu}{(1)_\nu(n - |\gamma|)_\nu}$$

$$< \frac{|\alpha|\ |\beta|}{n - |\gamma|} \sum_{\nu=1}^{\infty} \frac{(|\alpha| + 1)_\nu(|\beta| + 1)_\nu}{(1)_\nu(n + 1 - |\gamma|)_\nu};$$

the last series converges for $n > |\gamma| + |\alpha| + |\beta| - 1$ and is evidently a decreasing function of n, and since $1/(n - |\gamma|) \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} F(\alpha, \beta, \gamma + n; 1) = 1$. We therefore obtain the final result

$$\text{(70)} \quad F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \qquad \Re(\gamma - \alpha - \beta) > 0.$$

From this formula we shall now derive the relation

$$\text{(71)} \quad \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \qquad \Re(\alpha) > 0, \qquad \Re(\beta) > 0,$$

where the integral is known as Euler's integral of the first kind[15] Multiplying the binomial expansion, convergent for $|t| < 1$,

[15] J^{10} (for α and β real and positive).

$$(1-t)^{\beta-1} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(\beta-1)(\beta-2)\cdots(\beta-\nu)}{\nu!} t^{\nu} = \sum_{\nu=0}^{\infty} \frac{(\alpha)_{\nu}(1-\beta)_{\nu}}{(1)_{\nu}(\alpha)_{\nu}} t^{\nu}$$

by $t^{\alpha-1}$ and integrating from ϵ to s $(0 < \epsilon < s < 1)$, we obtain, since the series converges uniformly

$$\int_{\epsilon}^{s} t^{\alpha-1}(1-t)^{\beta-1}dt = \sum_{\nu=0}^{\infty} \frac{(\alpha)_{\nu}(1-\beta)_{\nu}}{(1)_{\nu}(\alpha)_{\nu}(\alpha+\nu)} (s^{\alpha+\nu} - \epsilon^{\alpha+\nu})$$

$$= \frac{s^{\alpha}}{\alpha} F(\alpha, 1-\beta, \alpha+1; s) - \frac{\epsilon^{\alpha}}{\alpha} F(\alpha, 1-\beta, \alpha+1; \epsilon).$$

Now let $\epsilon \to 0$ and $s \to 1$; since $\Re(\alpha) > 0$, $\Re(\beta) > 0$, the integral tends toward the left member of (71), $\epsilon^{\alpha} \to 0$ and on account of

$$\Re(\alpha + 1 - \alpha - (1-\beta)) = \Re(\beta) > 0,$$

$F(\alpha, 1-\beta, \alpha+1; 1)$ is convergent, so that Abel's theorem on power series gives $F(\alpha, 1-\beta, \alpha+1; s) \to F(\alpha, 1-\beta, \alpha+1; 1)$. Applying (70), we therefore obtain

$$\int_{0}^{1} t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{1}{\alpha} F(\alpha, 1-\beta, \alpha+1; 1) = \frac{1}{\alpha} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(1)\Gamma(\alpha+\beta)}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

We may now express the hypergeometric series as a definite integral. Let s have a fixed value such that $|s| < 1$, then $|st| \leqq |s|$ for $0 \leqq t \leqq 1$, and the binomial expansion

$$(1-st)^{-\beta} = \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu} t^{\nu}$$

converges uniformly in t for $0 \leqq t \leqq 1$. Assuming $\Re(\alpha) > 0$, $\Re(\gamma - \alpha) > 0$, multiplying by $t^{\alpha-1}(1-t)^{\gamma-\alpha-1}$ and integrating from 0 to 1, we may therefore integrate term by term on the right side and obtain

$$\int_{0}^{1} t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-st)^{-\beta}dt = \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu} \int_{0}^{1} t^{\alpha+\nu-1}(1-t)^{\gamma-\alpha-1}dt$$

$$= \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu} \cdot \frac{\Gamma(\alpha+\nu)\Gamma(\gamma-\alpha)}{\Gamma(\gamma+\nu)}$$

by (71), or since $\Gamma(\alpha+\nu) = (\alpha)_{\nu}\Gamma(\alpha)$ by $(J2)$,

$$\int_{0}^{1} t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-st)^{-\beta}dt = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma; s), \tag{72}$$

$$|s| < 1, \quad \Re(\alpha) > 0, \quad \Re(\gamma - \alpha) > 0.$$

12. **Application of Euler's summation formula to $\log \Gamma(s+a)$ and $\psi(s+a)$.** To secure the single-valuedness of the logarithms with which we shall deal in the following, we introduce a cut in the complex plane along the negative real axis (from 0 to $-\infty$) and subject the complex variable s to the restriction that it shall never cross this cut. Then every branch of the function $\log s$ is single-valued, and we shall define $\log s$ as that particular branch which is real for real and positive values of s. In particular, we have $\log (s)_n = \log s + \log (s+1) + \cdots + \log (s+n-1)$. We shall now apply (50) and (51) to the case $F(s) = \log s$, $b = 1$ and obtain first a proof that (s, n) approaches a limit as $n \to \infty$ for any s not on the negative real axis (that is, a new proof of the convergence of Euler's infinite product for $\Gamma(s)$, J3 and 3′, for these values of s) and second, Stirling's asymptotic series for $\log \Gamma(s)$.

Making $F(s) = \log s$, $b = 1$, $0 \leqq a < 1$ in (50), we find, since $\int_0^n \log (s+t)dt = (s+n)(\log (s+n) - 1) - s(\log s - 1)$ and $F'(s) = \frac{1}{s}$,

$$\log (s+a)_n = \sum_{\nu=0}^{n-1} \log (s+a+\nu) = (n+s+a-\tfrac{1}{2}) \log (s+n)$$
$$- n - (s+a-\tfrac{1}{2}) \log s - \int_0^n \frac{P_1(t-a)}{s+t} dt;$$

in particular, we find for $s = 1$, $a = 0$ and replacing n by $n-1$

$$\log (n-1)! = \sum_{\nu=0}^{n-2} \log (1+\nu) = (n-\tfrac{1}{2}) \log n - n + 1 - \int_0^{n-1} \frac{P_1(t)}{1+t} dt.$$

Adding $(s+a) \log n$ to the last equation, and subtracting the first, we obtain

$$\log (s+a, n) = \log (n-1)! + (s+a) \log n - \log (s+a)_n$$
$$= -(n+s+a-\tfrac{1}{2}) \log \left(1+\frac{s}{n}\right) + (s+a-\tfrac{1}{2}) \log s \tag{73}$$
$$+ 1 - \int_0^{n-1} \frac{P_1(t)}{1+t} dt + \int_0^n \frac{P_1(t-a)}{s+t} dt.$$

Since $P_1(t) = -\frac{dP_2(t)}{dt}$ by (41), an integration by parts gives

$$\int_0^n \frac{P_1(t-a)}{s+t} dt = \frac{P_2(a)}{s} - \frac{P_2(a)}{s+n} - \int_0^n \frac{P_2(t-a)}{(s+t)^2} dt,$$

and as $|P_2(t-a)| \leqq P_2(0)$ by (42), the integral $\int_0^\infty \frac{P_2(t-a)}{(s+t)^2} dt$ con-

verges for every s not on the negative real axis (theor. XVI cor.). Consequently

$$(74)\quad \lim_{n\to\infty}\int_0^n \frac{P_1(t-a)}{s+t}dt = \int_0^\infty \frac{P_1(t-a)}{s+t}dt = \frac{P_2(a)}{s} - \int_0^\infty \frac{P_2(t-a)}{(s+t)^2}dt,$$

and similarly $\lim_{n\to\infty}\int_0^{n-1}\frac{P_1(t)}{1+t}dt = \int_0^\infty \frac{P_1(t)}{1+t}dt$ exists; furthermore $\lim_{n\to\infty}(n+s+a-\frac{1}{2})\log\left(1+\frac{s}{n}\right) = \lim_{n\to\infty}(n+s+a-\frac{1}{2}).\frac{s}{n} = s$, and substitut ng all this in (73) we find

$$\lim_{n\to\infty}\log(s+a,n) = (s+a-\tfrac{1}{2})\log s - s + 1 - \int_0^\infty \frac{P_1(t)}{1+t}dt + \int_0^\infty \frac{P_1(t-a)}{s+t}dt.$$

From the existence of the limit to the left we infer at once that $\lim_{n\to\infty}(s+a,n) = \Gamma(s+a)$ exist for every s not on the cut, and that for such values of s

$$(75)\quad \log\Gamma(s+a) = (s+a-\tfrac{1}{2})\log s - s + k' + \int_0^\infty \frac{P_1(t-a)}{s+t}dt,$$

where we now only have to evaluate the constant $k' = 1 - \int_0^\infty \frac{P_1(t)}{1+t}dt$. To this purpose we observe that $\int_0^\infty \frac{P_1(t)}{n+t}dt = \int_{n-1}^\infty \frac{P_1(t)}{1+t}dt \to 0$ as $n\to\infty$ on account of the convergence of $\int_0^\infty \frac{P_1(t)}{1+t}dt$; making $a=0$, $s=n$ and $s=2n$ in (75), we obtain

$$2\log\Gamma(n) = (2n-1)\log n - 2n + 2k' + 2\int_0^\infty \frac{P_1(t)}{n+t}dt,$$

$$\log\Gamma(2n) = (2n-\tfrac{1}{2})\log n - 2n + k' + \int_0^\infty \frac{P_1(t)}{2n+t}dt,$$

and subtracting

$$k' + 2\int_0^\infty \frac{P_1(t)}{n+t}dt - \int_0^\infty \frac{P_1(t)}{2n+t}dt = \log\frac{\Gamma(n)^2 n^{\frac{1}{2}} 2^{2n-\frac{1}{2}}}{\Gamma(2n)}$$

$$= \log\frac{2^{\frac{1}{2}}\Gamma(n)n^{\frac{1}{2}}}{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)},$$

so that, as $n\to\infty$, $k' = \log(2^{\frac{1}{2}}\Gamma(\frac{1}{2})) = \log\sqrt{2\pi}$. This is essentially the method given in J § 4; another is the following: making $s = \frac{1}{2}\pm yi$, $y>0$ and $a=0$, so that for $t\geqq 0$, $|s+t|>y$ and $|s+t|>\frac{1}{2}+t$, $|s+t|^2 > y^{\frac{1}{2}}(\frac{1}{2}+t)^{\frac{3}{2}}$, it follows from (74) that

$$\left|\int_0^\infty \frac{P_1(t)}{\frac{1}{2} \pm yi + t}dt\right| < \frac{P_2(0)}{y} + \frac{P_2(0)}{y^{\frac{1}{2}}}\int_0^\infty \frac{dt}{(\frac{1}{2}+t)^{\frac{3}{2}}} \to 0 \text{ as } y \to \infty.$$

Now, by $(J6)$

$$\Gamma(\tfrac{1}{2} + yi)\Gamma(\tfrac{1}{2} - yi) = \frac{\pi}{\sin \pi(\frac{1}{2} + yi)} = \frac{2\pi i}{e^{\pi i(\frac{1}{2}+yi)} - e^{-\pi i(\frac{1}{2}+yi)}}$$

$$= \frac{2\pi}{e^{-\pi y} + e^{\pi y}} = \frac{2\pi e^{-\pi y}}{1 + e^{-2\pi y}},$$

and adding the two equations (75) for $a = 0$, $s = \frac{1}{2} + yi$ and $s = \frac{1}{2} - yi$,

$$\log 2\pi - \pi y - \log(1 + e^{-2\pi y})$$

$$= yi \log(\tfrac{1}{2} + yi) - (\tfrac{1}{2} + yi) - yi \log(\tfrac{1}{2} - yi) - (\tfrac{1}{2} - yi)$$

$$+ 2k' + \int_0^\infty \frac{P_1(t)}{\frac{1}{2} + yi + t}dt + \int_0^\infty \frac{P_1(t)}{\frac{1}{2} - yi + t}dt.$$

But $\log(\frac{1}{2} + yi) = \log y + \log i + \log\left(1 + \frac{1}{2yi}\right) = \log y + \frac{\pi i}{2} + \frac{1}{2yi} + \cdots$, and similarly $\log(\frac{1}{2} - yi) = \log y - \frac{\pi i}{2} - \frac{1}{2yi} + \cdots$; introducing this in the preceding equation and letting $y \to \infty$, we find $2k' = \log 2\pi$. Thus (75) finally becomes [16]

$$\log \Gamma(s + a) = (s + a - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} + \omega(s, a),$$

$$\omega(s, a) = \int_0^\infty \frac{P_1(t - a)}{s + t}dt. \tag{76}$$

When $a = 0$,

$$\omega(s, 0) = \int_0^\infty \frac{P_1(t)}{s + t}dt = \sum_{\nu=0}^\infty \int_\nu^{\nu+1} \frac{P_1(t)}{s + t}dt$$

$$= \sum_{\nu=0}^\infty \int_0^1 \frac{P_1(t)}{s + \nu + t}dt = \sum_{\nu=0}^\infty \int_0^1 \frac{\frac{1}{2} - t}{s + \nu + t}dt,$$

and performing the integrations in the last expression, we obtain Gudermann's formula for $\omega(s, 0)$, $(J\ 13)$.

In (76), we now integrate the expression for $\omega(s, a)$ by parts; the result may be read off immediately from a comparison of (50) and (51), observing that $F^k(s) = (-1)^{k-1}(k-1)!/s^k$. Thus we obtain

[16] For $a = 0$ and s real, this formula is due to Gilbert, Ph. (J^{22}), and for $a = 0$ and s complex to Stieltjes, T. J., Sur le développement de log $\Gamma(a)$, Journn. de Math., ser. 4, vol. 5 (1889), pp. 425–444. For $0 \leqq a < 1$ and s complex, see Lindelöf, E., Le calcul des résidus, Paris, 1905, pp. 95–97.

$$(77)\qquad \omega(s, a) = \sum_{k=1}^{2m-1} (-1)^{[k(k-1)]/2} \frac{(k-1)!P_{k+1}(a)}{s^k} + (-1)^m(2m-1)! \int_0^\infty \frac{P_{2m}(t-a)}{(s+t)^{2m}}\, dt.$$

To form an estimate of the remainder, let $s = |s| e^{\theta i}$, $-\pi < \theta < \pi$; then

$$|s+t|^2 = (|s| \cos\theta + t)^2 + |s|^2 \sin^2\theta$$

$$= (|s| + t)^2 \cos^2 \frac{\theta}{2} + (|s| - t)^2 \sin^2 \frac{\theta}{2} \geqq (|s| + t)^2 \cos^2 \frac{\theta}{2},$$

and since $|P_{2m}(t-a)| \leqq P_{2m}(0)$, we obtain

$$\left| (-1)^m(2m-1)! \int_0^\infty \frac{P_{2m}(t-a)}{(s+t)^{2m}}\, dt \right| < \frac{(2m-1)!P_{2m}(0)}{\cos^{2m} \theta/2} \int_0^\infty \frac{dt}{(|s| + t)^{2m}}$$

$$= \frac{(2m-2)!P_{2m}(0)}{\cos^{2m} \theta/2 \cdot |s|^{2m-1}}.$$

Joining the term with $k = 2m - 1$ in (77) to the remainder, we may therefore write

$$(78)\qquad \omega(s, a) = \sum_{k=1}^{2m-2} (-1)^{[k(k-1)]/2} \frac{(k-1)!P_{k+1}(a)}{s^k} + \frac{(2m-2)!P_{2m}(0)\cdot h}{\cos^{2m} \theta/2 \cdot |s|^{2m-1}}, \qquad |h| < 2.$$

In the particular cases where $a = 0$ or $a = \frac{1}{2}$, $P_{k+1}(a) = 0$ for $k + 1$ odd, and therefore, replacing k by $2k - 1$, (78) becomes

$$(79)\qquad \omega(s, a) = \sum_{k=1}^{m-1} (-1)^{k-1} \frac{(2k-2)!P_{2k}(a)}{s^{2k+1}} + \frac{(2m-2)!P_{2m}(0)\cdot h}{\cos^{2m} \theta/2 \cdot |s|^{2m-1}}, \qquad |h| < 2, \qquad a = 0 \text{ or } \tfrac{1}{2}.$$

When s is real and positive, we may use the much more accurate remainder terms (56) and (60). In fact, for $a = 0$ and $\alpha = -1$,

$$\alpha F^{(2m)}(s+t) = \frac{(2m-1)!}{(s+t)^{2m}}$$

is positive and decreases as t increases from 0 to ∞, and

$$\alpha(F^{(2m)}(s+\nu+t) + F^{(2m)}(s+\nu+1-t))$$

$$= \frac{(2m-1)!}{(s+\nu+t)^{2m}} + \frac{(2m-1)!}{(s+\nu+1-t)^{2m}}$$

is positive and decreases as t increases from 0 to $\frac{1}{2}$ (since the derivative

in respect to t is negative). Similarly we see that for $a = \frac{1}{2}$ and $\alpha = -1$, the conditions for the applicability of (60) are satisfied. Taking the values of $P_{2k}(0)$ and $P_{2k}(\frac{1}{2})$ from (48), we therefore obtain[17]

$$\omega(s, 0) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}B_k}{(2k-1)\cdot 2k} \cdot \frac{1}{s^{2k-1}} + \frac{(-1)^{m-1}B_m}{(2m-1)\cdot 2m} \frac{1}{(s+h)^{2m-1}}, \quad 0 < h < \tfrac{1}{2}, \quad s > 0. \tag{80}$$

$$\begin{aligned}\omega(s, \tfrac{1}{2}) = {} & \sum_{k=1}^{m-1} \left(1 - \frac{1}{2^{2k-1}}\right) \frac{(-1)^k B_k}{(2k-1)\cdot 2k} \cdot \frac{1}{s^{2k-1}} \\ & + \left(1 - \frac{1}{2^{2m-1}}\right) \frac{(-1)^m B_m}{(2m-1)\cdot 2m} \frac{1}{(s+h)^{2m-1}}, \quad 0 < h < \tfrac{1}{2}, \quad s > 0.\end{aligned} \tag{81}$$

The infinite series

$$\sum_{k=1}^{\infty} (-1)^{[k(k-1)]/2} \frac{(k-1)!P_{k+1}(a)}{s^k},$$

the first $2m - 2$ terms of which occur in (78), does not converge for any value of s. First suppose $a \neq 0$ and $a \neq \frac{1}{2}$, then the ratio of the terms for which $k = 2m + 2$ and $k = 2m$ may be written, using the expressions (42) for $P_{2m+3}(a)$ and $P_{2m+1}(a)$:

$$-\frac{m(2m+1)}{2\pi^2 s^2} \cdot \frac{\displaystyle\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m+2}}}{\displaystyle\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}}},$$

and since $\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}}$ converges uniformly in respect to m for $m \geqq 1$, we have $\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}} \to \sin 2\pi a$ as $m \to \infty$, so that the second factor in our ratio approaches unity as $m \to \infty$. The first factor, however, approaches infinity, and consequently the series cannot converge. When $a = 0$ or $a = \frac{1}{2}$, a similar argument applies to the ratio of the terms with $k = 2m + 3$ and $k = 2m + 1$.

Proceeding to the logarithmic derivative of the Gamma function, we find, making $F(s) = \frac{1}{s}$, $b = 1$, $0 \leqq a < 1$ in (50), that

$$\sum_{\nu=0}^{n-1} \frac{1}{s + a + \nu} = \log(s+n) - \log s + (\tfrac{1}{2} - a)\left(\frac{1}{s} - \frac{1}{s+n}\right) + \int_0^n \frac{P_1(t-a)dt}{(s+t)^2},$$

and since the last integral converges as $n \to \infty$, and $\psi(s + a) = \lim_{n\to\infty} \left(\log n - \sum_{\nu=0}^{n-1} \frac{1}{s+a+\nu}\right)$ by (J 19′), it is seen that

[17] (80) is due to Stirling (J[21]) and (81) to Gauss (J[11]), both without remainder terms. The form of the remainder term given in (80) is due to Sonin.[9] For $a = 0$, the remainder term in (79) was given by Stieltjes[16] (who in this particular case obtains $|h| \leqq 1$).

$$\psi(s + a) = \log s - \omega^*(s, a)$$

(82)
$$\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} + \int_0^\infty \frac{P_1(t - a)dt}{(s + t)^2}.$$

Treating the integral in the same manner as the corresponding one for $\omega(s, a)$, we readily obtain

(83)
$$\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} + \sum_{k=2}^{2m}(-1)^{[(k-1)(k-2)]/2}\frac{(k-1)!P_k(a)}{s^k} + (-1)^m(2m)!\int_0^\infty \frac{P_{2m}(t - a)dt}{(s + t)^{2m+1}},$$

(84)
$$\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} + \sum_{k=2}^{2m-1}(-1)^{[(k-1)(k-2)]/2}\frac{(k-1)!P_k(a)}{s^k} + \frac{(2m-1)!P_{2m}(0)\cdot h}{\cos^{2m}\theta/2\cdot|s|^{2m}}, \qquad |h| < 2,$$

and for s real and positive

(85)
$$\omega^*(s, 0) = \frac{1}{2s} + \sum_{k=1}^{m-1}\frac{(-1)^{k-1}B_k}{2k}\cdot\frac{1}{s^{2k}} + \frac{(-1)^{m-1}B_m}{2m}\cdot\frac{1}{(s + h)^{2m}}, \qquad 0 < h < \tfrac{1}{2},$$

(86)
$$\omega^*(s, \tfrac{1}{2}) = \sum_{k=1}^{m-1}\left(1 - \frac{1}{2^{2k-1}}\right)\frac{(-1)^k B_k}{2k}\cdot\frac{1}{s^{2k}} + \left(1 - \frac{1}{2^{2m-1}}\right)\frac{(-1)^m B_m}{2m}\cdot\frac{1}{(s + h)^{2m}}, \qquad 0 < h < \tfrac{1}{2};$$

furthermore, the same argument as before shows the divergence of the infinite series of which the initial terms occur in (84).

A divergent series

$$a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_n}{s^n} + \cdots,$$

in which the sum of the $n + 1$ first terms is $S_n(s)$, is said to be an asymptotic expansion of a function $f(s)$ for a given range of values of $\theta = \text{arc } s$, if the expansion

$$R_n(s) = s^n(f(s) - S_n(s))$$

satisfies the condition $\lim\limits_{|s|\to\infty} R_n(s) = 0$ for n fixed, even though $\lim\limits_{n\to\infty} |R_n(s)| = \infty$ when s is fixed. In this sense, (78) and (79) are asymptotic expansions of $\omega(s, a)$, and (84) of $\omega^*(s, a)$, for $-\pi + \epsilon \leqq \theta \leqq \pi - \epsilon$, where ϵ is as small as we please, i. e., for s inside of any infinite sector not containing the negative real axis.

It is clear from the definition that a function $f(s)$ cannot have two different asymptotic expansions $a_0 + \frac{a_1}{s} + \cdots$ and $b_0 + \frac{b_1}{s} + \cdots$ both valid in the same sector (or even for a single common value of $\theta = \operatorname{arc} s$), since then we should have

$$\lim_{|s| \to \infty} s^n \left(a_0 - b_0 + \frac{a_1 - b_1}{s} + \cdots + \frac{a_n - b_n}{s^n} \right) = 0,$$

whence $a_0 = b_0$, $a_1 = b_1$, $\cdots$. This remark will be used in the following to identify various expressions for the coefficients in (78) and (84).

13. Integrals for $\log \Gamma(s)$ **and** $\psi(s)$ **of Cauchy and Gauss.** We begin by showing that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a} \quad \text{for } \Re(a) > 0,\ \Re(b) > 0, \tag{87}$$

where $\log \frac{b}{a}$ is the principal logarithm. By direct integration we find, for $x \geqq \epsilon > 0$,

$$\int_0^T e^{-(x+iy)t} dt = \frac{1}{x+iy} (1 - e^{-(x+iy)T}),$$

and since $\left| \frac{e^{-(x+iy)T}}{x+iy} \right| \leqq \frac{e^{-xT}}{x} \leqq \frac{e^{-\epsilon T}}{\epsilon} \to 0$ as $T \to \infty$, it follows that

$$\int_0^\infty e^{-(x+iy)t} dt = \frac{1}{x+iy}, \tag{88}$$

the integral being uniformly convergent for $x \geqq \epsilon$. Let $\Re(a) \geqq \epsilon$, $\Re(b) \geqq \epsilon$, and write $x + iy = a(1-u) + bu$ where $0 \leqq u \leqq 1$, then $x \geqq \epsilon$, and from the uniform convergence of our integral it follows, by theor. XX, that

$$\begin{aligned} \int_0^1 \frac{(b-a)du}{a + (b-a)u} &= \int_0^1 du \int_0^\infty (b-a) e^{-(a+(b-a)u)t} dt \\ &= \int_0^\infty dt \int_0^1 (b-a) e^{-(a+(b-a)u)t} du \\ &= \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt. \end{aligned}$$

But the first single integral is found by direct integration to equal one of the values of $\log \frac{b}{a}$, and since the real part of the denominator in the integrand is not less than ϵ for $\Re(a) \geqq \epsilon$, $\Re(b) \geqq \epsilon$, it follows that the

integral is continuous for such values of a and b. Since the integral is obviously real for real values of a and b, it follows that the value of $\log \frac{a}{b}$ in question must be the principal one, and (87) is proved. In the equation (J 19″), making $a = 1$,

$$\psi(s) = \lim_{n\to\infty}\left(\log(n+1) - \sum_{\nu=0}^{n-1}\frac{1}{s+\nu}\right)$$

we now assume $\Re(s) > 0$; then we have, by (88) and (87), for $\nu = 0, 1, \cdots$,

$$\frac{1}{s+\nu} = \int_0^\infty e^{-(s+\nu)t}dt, \qquad \log(n+1) = \int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{t}dt,$$

and consequently

$$\psi(s) = \lim_{n\to\infty}\int_0^\infty\left(\frac{e^{-t} - e^{-(n+1)t}}{t} - \sum_{\nu=0}^{n-1}e^{-(s+\nu)t}\right)dt.$$

Upon summation of the finite geometric series to the right, this becomes

$$\psi(s) = \lim_{n\to\infty}\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)(1 - e^{-nt})dt.$$

Now let $\Re(s) \geqq \epsilon$ and $|s| < A$, where $\epsilon < 1$ is as small and A as large as we please; using the power series for e^{-t} and e^{-st}, we see that the first factor under the integral sign is expansible in a power series in t, convergent for $|t| < r$, where r is independent of s, and consequently

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| < M_1 = M_1(\epsilon, A)$$

for $0 \leqq t < r$. On the other hand, we have for $t \geqq r$, since $\epsilon < 1$

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| \leqq \frac{e^{-t}}{r} + \frac{e^{-\epsilon t}}{1-e^{-r}} < \left(\frac{1}{r} + \frac{1}{1-e^{-r}}\right)e^{-\epsilon t},$$

so that finally, for $t \geqq 0$ and $\Re(s) \geqq \epsilon$, $|s| \leqq A$

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| < M(\epsilon, A)\cdot e^{-\epsilon t}.$$

Consequently the integral

$$\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)dt$$

converges uniformly in respect to s for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, and

$$\left|\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)e^{-nt}dt\right| < M(\epsilon, A)\int_0^\infty e^{-(\epsilon+n)t}dt = \frac{M(\epsilon, A)}{\epsilon+n} \to 0$$

as $n \to \infty$, so that finally

$$\psi(s) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}}\right) dt, \qquad \Re(s) > 0. \tag{89}$$

We may transform this integral by writing $t = \log(1 + u)$, $\Delta = e^\delta - 1$,

$$\psi(s) = \lim_{\delta \to 0} \int_\delta^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}}\right) dt = \lim_{\delta \to 0} \left(\int_\delta^\infty \frac{e^{-t}}{t} dt - \int_\Delta^\infty \frac{du}{u(1 + u)^s}\right)$$

and since

$$0 < \int_\delta^\Delta \frac{e^{-t}}{t} dt < \int_\delta^\Delta \frac{dt}{t} = \log \frac{e^\delta - 1}{\delta} \to 0 \text{ as } \delta \to 0,$$

this may be written

$$\psi(s) = \lim_{\delta \to 0} \left(\int_\Delta^\infty \frac{e^{-t}}{t} dt - \int_\Delta^\infty \frac{du}{u(1 + u)^s}\right),$$

or finally, replacing u by t in the last integral,

$$\psi(s) = \int_0^\infty (e^{-t} - (1 + t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0. \tag{90}$$

Again, replacing e^{-t} by t in (89), we obtain

$$\psi(s) = \int_0^1 \left(-\frac{1}{\log t} - \frac{t^{s-1}}{1 - t}\right) dt, \qquad \Re(s) > 0. \tag{91}$$

For $s = 1$, we have $\psi(s) = -C$, whence by (89), (90), (91)

$$\begin{aligned} C &= \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t}\right) e^{-t} dt \\ &= \int_0^\infty \left(\frac{1}{1 + t} - e^{-t}\right) \frac{dt}{t} \\ &= \int_0^1 \left(\frac{1}{1 - t} + \frac{1}{\log t}\right) dt, \end{aligned} \tag{92}$$

and consequently

$$\begin{aligned} \psi(s) + C &= \int_0^\infty \frac{e^{-t} - e^{-st}}{1 - e^{-t}} dt \\ &= \int_0^\infty ((1 + t)^{-1} - (1 + t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0 \\ &= \int_0^1 \frac{1 - t^{s-1}}{1 - t} dt. \end{aligned} \tag{93}$$

Since (89) converges uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, the corresponding integral for $\log \Gamma(s)$ may be obtained by integration according to theor.

XX, but it is equally simple to start from (J 3″) for $a = 1$, or rather the equivalent formula

$$\log \Gamma(s) = \lim_{n\to\infty} \left[(s-1) \log (n+1) - \sum_{\nu=1}^{n} \log \frac{s+\nu-1}{\nu} \right].$$

For $\Re(s) > 0$, (87) gives

$$\log \frac{s+\nu-1}{\nu} = \int_0^\infty \frac{e^{-\nu t} - e^{-(s+\nu-1)t}}{t} dt, \quad \log (n+1) = \int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{t} dt,$$

$$\log \Gamma(s) = \lim_{n\to\infty} \int_0^\infty \left[(s-1)(e^{-t} - e^{-(n+1)t}) - \sum_{\nu=1}^{n} (e^{-\nu t} - e^{-(s+\nu-1)t}) \right] \frac{dt}{t}$$

$$= \lim_{n\to\infty} \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1-e^{-t}} \right] (1 - e^{-nt}) \frac{dt}{t}.$$

When $\Re(s) \geqq \epsilon$ $(\epsilon < 1)$, $|s| \leqq A$, it is seen exactly as before that

$$\left| \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1-e^{-t}} \right] \frac{1}{t} \right| < M(\epsilon, A) e^{-\epsilon t}$$

for $t \geqq 0$, where M depends only on ϵ and A but not on s, and that

$$(94) \qquad \log \Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1-e^{-t}} \right] \frac{dt}{t}, \qquad \Re(s) > 0,$$

the integral being uniformly convergent for $\Re(s) \geqq \epsilon$, $|s| \leqq A$. In the same manner as (90) was derived from (89), we also find

$$(95) \quad \log \Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{(1+t)^{-s} - (1+t)^{-1}}{\log (1+t)} \right] \frac{dt}{t}, \quad \Re(s) > 0.$$ [18]

14. Integrals of Raabe and Binet. The integral of Raabe[19]

$$(96) \quad \int_0^1 \log \Gamma(s+a) da = s \log s - s + \log \sqrt{2\pi}, \qquad (s \text{ not on cut}),$$

results most readily from (76), which gives upon integration

$$\int_0^1 \log \Gamma(s+a) da = s \log s - s + \log \sqrt{2\pi} + \int_0^1 da \int_0^\infty \frac{P_1(t-a)dt}{s+t}.$$

Here the double integral vanishes, being the limit as $n \to \infty$ of

[18] Eq. (89) and (91) are due to Gauss (J[11]), (90) to Dirichlet, P. G. L., Sur les intégrales Eulériennes, Journ. f. Math., vol. 15 (1836), pp. 258–263 = Werke, vol. 1, pp. 271–278. The last of (93) was given by Legendre, A. M., Exercices de calcul intégral, vol. 2 (1817), p. 45, and (94) and (95) by Cauchy, A. L., Exercices d'analyse et de physique mathématique, vol. 2 (Paris, 1841), p. 380.

[19] Raabe, J. L., Angenäherte Bestimmung der Factorenfolge, etc., Journ. f. Math., vol. 25 (1843), pp. 146–159.

$$\int_0^1 da \int_0^n \frac{P_1(t-a)dt}{s+t} = \int_0^n \frac{dt}{s+t}\int_0^1 P_1(t-a)da,$$

and by (41), $\int_0^1 P_1(t-a)da = P_2(t-1) - P_2(t) = 0$. Another proof may be derived by means of $(J\ 3')$:

$$\int_0^1 \log\Gamma(s+a)da = \lim_{n\to\infty}\int_0^1\left[\log\Gamma(n)+(s+a)\log n - \sum_{\nu=0}^{n-1}\log(s+a+\nu)\right]da$$

$$= \lim_{n\to\infty}\left[\log\Gamma(n) + (s+\tfrac{1}{2})\log n - \int_0^n \log(s+a)da\right]$$

$$= \lim_{n\to\infty}[\log\Gamma(n) + (s+\tfrac{1}{2})\log n - (s+n)\log(s+n) + s\log s + n]$$

$$= s\log s - s + \lim_{n\to\infty}[\log\Gamma(n) - (n-\tfrac{1}{2})\log n + n],$$

and the last limit is $\log\sqrt{2\pi}$, as is seen from Stirling's formula. We shall now derive an expression for $\omega(s, a)$ similar to the integral for $\log\Gamma(s)$ in (94). From (76) we obtain, since $P_1(x)$ has the period unity,

$$\omega(s, a) = \lim_{n\to\infty}\int_0^n \frac{P_1(u-a)}{s+u}du = \lim_{n\to\infty}\sum_{\nu=0}^{n-1}\int_0^1 \frac{P_1(u-a)}{s+u+\nu}du,$$

and from (88), for $\Re(s) > 0$,

$$\frac{1}{s+u+\nu} = \int_0^\infty e^{-(s+u+\nu)t}dt,$$

so that

$$\omega(s, a) = \lim_{n\to\infty}\sum_{\nu=0}^{n-1}\int_0^1 du\int_0^\infty P_1(u-a)e^{-(s+u+\nu)t}dt.$$

By theorem XX, the order of integration in the repeated integrals may be inverted, and since $P_1(x) = -\frac{1}{2} - x$ for $-1 < x < 0$ but $P_1(x) = \frac{1}{2} - x$ for $0 < x < 1$, we have

$$\int_0^1 P_1(u-a)e^{-ut}du = \int_0^a (a-\tfrac{1}{2}-u)e^{-tu}du + \int_a^1 (a+\tfrac{1}{2}-u)e^{-ut}du$$

$$= \frac{1}{t}\left[\left(a-\tfrac{1}{2}-\frac{1}{t}\right)(1-e^{-t}) + e^{-at}\right],$$

as is readily seen upon integration by parts, so that

$$\omega(s, a) = \lim_{n\to\infty}\sum_{\nu=0}^{n-1}\int_0^\infty\left[\left(a-\tfrac{1}{2}-\frac{1}{t}\right)(1-e^{-t}) + e^{-at}\right]e^{-(s+\nu)t}\frac{dt}{t}$$

$$= \lim_{n\to\infty}\int_0^\infty\left[a-\tfrac{1}{2}-\frac{1}{t}+\frac{e^{-at}}{1-e^{-t}}\right](1-e^{-nt})e^{-st}\frac{dt}{t}.$$

Exactly as in the proof of (89), we see that for $t \geqq 0$,

$$\left| \left[a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right] \cdot \frac{1}{t} \right| < M$$

and that consequently, for $\Re(s) > 0$, the limiting process may be performed under the integral sign, so that finally[20]

$$(97) \qquad \omega(s, a) = \int_0^\infty \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right) \frac{e^{-st}dt}{t}, \qquad \Re(s) > 0.$$

Writing

$$(98) \qquad f(t, a) = \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right) \cdot \frac{1}{t},$$

this function is holomorphic in the neighborhood of $t = 0$, and consequently $\dfrac{\partial^k f(t, a)}{\partial t^k}$ is bounded in this neighborhood. Using Leibniz's formula for obtaining this derivative, each term will contain a negative power of t as factor, and consequently $\dfrac{\partial^k f(t, a)}{\partial t^k} \to 0$ as $t \to \infty$. We therefore have

$$\left| \frac{\partial^k f(t, a)}{\partial t^k} \right| < M_k \text{ for } t \geqq 0,$$

where M_k is independent of t. Now integrating by parts in (97), we obtain

$$\omega(s, a) = \left[- \sum_{k=1}^{2m-2} \frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \frac{e^{-st}}{s^k} \right]_{t=0}^{t=\infty} + \frac{1}{s^{2m-2}} \int_0^\infty \frac{\partial^{2m-2} f(t, a)}{\partial t^{2m-2}} e^{-st}dt$$

and since $\Re(s) = |s| \cos \theta > 0$ and the partial derivatives of $f(t, a)$ are bounded, each term in the sum vanishes at the upper limit and

$$\left| \int_0^\infty \frac{\partial^{2m-2} f(t, a)}{\partial t^{2m-2}} e^{-st}dt \right| < \int_0^\infty M_{2n-2} e^{-|s| \cos \theta \cdot t} dt = \frac{M_{2m-2}}{|s| \cos \theta},$$

we see that

$$(99) \quad \omega(s, a) = \sum_{k=1}^{2m-1} \left[\frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \right]_{t=0} \cdot \frac{1}{s^k} + \frac{M_{2m-2} \cdot h}{|s|^{2m-1} \cos \theta}, \qquad |h| < 1$$

Consequently (99) is an asymptotic expansion of $\omega(s, a)$ for $\Re(s) > 0$, and must therefore coincide with (78), whence

$$\left[\frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \right]_{t=0} = (-1)^{[k(k-1)]/2} (k-1)! P_{k+1}(a), \qquad k = 1, 2, \cdots.$$

Using this in the expansion of $f(t, a)$ by Taylor's theorem, we obtain a formula which is essentially identical with (49).

[20] For $a = 0$, this expression is due to Binet (J[24]).

It should be noted that this method of deriving the asymptotic expansion of $\omega(s, a)$ only shows the expansion to be valid in the half-plane $\Re(s) > 0$, whereas the method of § 12 is valid when s is not on the negative real axis.

15. **The generalized Schaar integral.** To obtain this integral, we shall replace each term in the expression (J 24)

$$\frac{d\psi(s)}{ds} = \sum_{\nu=0}^{\infty} \frac{1}{(s+\nu)^2}$$

by an integral over a rational function (instead of the exponential used in § 13). Let $s = x + yi$ and $x > 0$; then

$$\begin{aligned}\int_0^\infty \frac{sdt}{t^2+s^2} &= \int_0^\infty \frac{i}{2}\left(\frac{1}{t-y+xi} - \frac{1}{t+y-xi}\right)dt \\ &= \frac{i}{2}\int_0^\infty \left(\frac{t-y-xi}{(t-y)^2+x^2} - \frac{t+y+xi}{(t+y)^2+x^2}\right)dt \\ &= \frac{i}{2}\left[\frac{1}{2}\log\frac{(t-y)^2+x^2}{(t+y)^2+x^2} - i\arctan\frac{t-y}{x} - i\arctan\frac{t+y}{x}\right]_{t=0}^{t=\infty} \\ &= \frac{\pi}{2}.\end{aligned}$$

Now let $\sigma = \xi + \eta i$ and $\xi > 0$, then

$$\frac{\pi}{2}(s - \sigma) = \int_0^\infty \left(\frac{s^2}{t^2+s^2} - \frac{\sigma^2}{t^2+\sigma^2}\right)dt,$$

whence

$$\frac{1}{s+\sigma} = \frac{2}{\pi}\int_0^\infty \frac{t^2dt}{(t^2+s^2)(t^2+\sigma^2)}, \qquad \Re(s) > 0, \qquad \Re(\sigma) > 0,$$

and this is also true for $\sigma = 0$, since the integral then reduces to the preceding one. Differentiating in respect to x, we obtain

$$(100) \quad \frac{1}{(s+\sigma)^2} = \frac{2}{\pi}\int_0^\infty \frac{2st^2dt}{(t^2+s^2)^2(t^2+\sigma^2)}, \qquad \Re(s) > 0, \qquad \Re(\sigma) > 0,$$

and this is also true for $\sigma = 0$. In fact we have [21]

$$(101) \quad |t^2 + s^2| \geqq (t^2 + |s|^2)\cos\theta \text{ for } t \geqq 0, \qquad s = |s|e^{\theta i}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

and for $s = |s|e^{\theta i}$, $\sigma = |\sigma|e^{\phi i}$, $-\frac{\pi}{2} + \epsilon \leqq \theta$, $\varphi \leqq \frac{\pi}{2} - \epsilon$, $A \geqq |s| \geqq \delta$, $|\sigma| \geqq \delta$, the integrand in (100) is consequently less than

[21] Since $|t^2 + s^2|^2 = |t^2 + |s|^2e^{2\theta i}|^2 = (t^2 + |s|^2\cos 2\theta)^2 + (|s|^2\sin 2\theta)^2 = (t^2 + |s|^2)^2\cos^2\theta + (t^2 - |s|^2)^2\sin^2\theta$.

$$\frac{2At^2}{\sin^3\epsilon(t^2+\delta^2)^3},$$

so that the integral (100) is uniformly convergent in respect to s in the region considered, and theorem XXI is applicable; a similar proof applies to the case $\sigma = 0$. In order to obtain a formula applying to all values of s except those on the negative real axis, let us assume $-\pi+2\epsilon \leqq \theta \leqq \pi - 2\epsilon$; then we may determine an angle α such that $-\frac{\pi}{2}+\epsilon \leqq \alpha \leqq \frac{\pi}{2} - \epsilon$ and $-\frac{\pi}{2}+\epsilon \leqq \theta - \alpha \leqq \frac{\pi}{2} - \epsilon$. In (100) now replace s by $se^{-\alpha i}$ and σ by $\nu e^{-\alpha i}$, where ν is a positive integer or zero, and form the sum from $\nu = 0$ to $\nu = n-1$; we obtain

$$\sum_{\nu=0}^{n-1}\frac{1}{(s+\nu)^2} = \frac{2}{\pi}\int_0^\infty \frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i}+s^2)^2}\sum_{\nu=0}^{n-1}\frac{1}{t^2e^{2\alpha i}+\nu^2}\,dt.$$

But by (101), $|t^2e^{2\alpha i}+\nu^2| \geqq (t^2+\nu^2)\cos\alpha \geqq \nu^2\sin\epsilon$, and for $|s| \geqq \delta$,

$$(102)\qquad \begin{aligned} |t^2e^{2\alpha i}+s^2| = |t^2+s^2e^{-2\alpha i}| &\geqq (t^2+|s|^2)\cos(\theta-\alpha) \\ &\geqq (t^2+\delta^2)\sin\epsilon; \end{aligned}$$

consequently

$$\left|\int_0^\infty \frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i}+s^2)^2}\sum_{\nu=n}^{\infty}\frac{1}{t^2e^{2\alpha i}+\nu^2}\,dt\right| \leqq \frac{2|s|}{\sin^3\epsilon}\sum_{\nu=n}^{\infty}\frac{1}{\nu^2}\cdot\int_0^\infty\frac{t^2dt}{(t^2+\delta^2)^2} \to 0$$

as $n\to\infty$, since the last integral converges and $\Sigma\frac{1}{\nu^2}$ is a convergent series; therefore

$$\psi'(s) = \sum_0^\infty\frac{1}{(s+\nu)^2} = \frac{2}{\pi}\int_0^\infty\frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i}+s^2)^2}\sum_{\nu=0}^\infty\frac{1}{t^2e^{2\alpha i}+\nu^2}\,dt.$$

But from the partial fraction expansion of the cotangent it follows (see § 8) that

$$\sum_{\nu=0}^\infty\frac{2t}{t^2+\nu^2} = \frac{2\pi}{e^{2\pi t}-1}+\frac{1}{t}+\pi,$$

and changing t into $te^{\alpha i}$, we find

$$\psi'(s) = \frac{2}{\pi}\int_0^\infty\frac{se^{\alpha i}dt}{(t^2e^{2\alpha i}+s^2)^2} + 2\int_0^\infty\frac{se^{2\alpha i}tdt}{(t^2e^{2\alpha i}+s^2)^2} + 4\int_0^\infty\frac{se^{2\alpha i}t}{(t^2e^{2\alpha i}+s^2)^2}\frac{dt}{e^{2\pi te^{\alpha i}}-1}.$$

The first of these integrals is found, by (100) for $\sigma = 0$, to equal $1/2s^2$, the value $1/s$ of the second is obtained by direct integration, and the third converges uniformly for $A \geqq |s| \geqq \delta$, as is seen from (100) and the fact that

$$(103) \qquad \left|\frac{t}{e^{2\pi t e^{\alpha i}}-1}\right| < \begin{cases} \dfrac{1}{\sin\epsilon} \text{ for } 0 < t < 1, \\[2ex] \dfrac{1}{t^2 \sin^3\epsilon} \text{ for } t > 1.^{22} \end{cases}$$

Hence the application of theorem XXI to the third integral gives

$$\psi'(s) = \frac{1}{s} + \frac{1}{2s^2} - 2\frac{\partial}{\partial x}\int_0^\infty \frac{e^{2\alpha i}}{t^2e^{2\alpha i}+s^2}\cdot\frac{tdt}{e^{2\pi t e^{\alpha i}}-1},$$

and integration in respect to x,

$$\psi(s) = \log s - \frac{1}{2s} + c(y) - 2\int_0^\infty \frac{e^{2\alpha i}}{t^2e^{2\alpha i}+s^2}\cdot\frac{tdt}{e^{2\pi t e^{\alpha i}}-1},$$

where we have denoted the integration constant by $c(y)$. Now in $s = x + yi$ let $x \to \infty$; then $\theta = \arctan y/x \to 0$, so that for any α where $|\alpha| \leqq \pi/2 - \epsilon$ and α has the same sign as y, we have $|\theta - \alpha| \leqq \pi/2 - \epsilon$ for x sufficiently large. Then, by (102) and (103)

$$\left|\int_0^\infty \frac{e^{2\alpha i}}{t^2e^{2\alpha i}+s^2}\cdot\frac{tdt}{e^{2\pi t e^{\alpha i}}-1}\right| < \frac{1}{|s|^2\sin\epsilon}\left(\int_0^1\frac{dt}{\sin\epsilon} + \int_1^\infty\frac{dt}{t^2\sin^3\epsilon}\right) \to 0$$

as $x \to \infty$, and since also $\psi(s) - \log s \to 0$ by (84), it follows that $c(y) = 0$ for all values of y. Consequently

$$(104) \qquad \psi(s) = \log s - \frac{1}{2s} - 2\int_0^\infty \frac{e^{2\alpha i}}{t^2e^{2\alpha i}+s^2}\cdot\frac{tdt}{e^{2\pi t e^{\alpha i}}-1}, \qquad |\theta| < \pi, \quad |\theta| < \frac{\pi}{2}, \quad |\theta - \alpha| < \frac{\pi}{2},$$

and upon integration by parts

$$(105) \qquad \psi(s) = \log s - \frac{1}{2s} + \frac{1}{\pi}\int_0^\infty \frac{e^{\alpha i}(s^2 - t^2e^{2\alpha i})}{(t^2e^{2\alpha i}+s^2)^2}\log\left(1 - e^{-2\pi t e^{\alpha i}}\right)dt.$$

Since $|\alpha| < \pi/2$, we have $|e^{-2\pi t e^{\alpha i}}| = e^{-2\pi t\cos\alpha} < 1$ for $t > 0$, and consequently

$$(106) \qquad \left|\log\left(1 - e^{-2\pi t e^{\alpha i}}\right)\right| = \left|\sum_{\nu=1}^\infty \frac{e^{-2\pi t e^{\alpha i}}}{\nu}\right| \leqq \sum_{\nu=1}^\infty \frac{e^{-2\pi t\cos\alpha}}{\nu} = -\log\left(1 - e^{-2\pi t\cos\alpha}\right);$$

the last two expressions show that $\log(1 - e^{-2\pi t e^{\alpha i}})$ becomes infinite as $\log t$ for $t \to 0$ and vanishes as $e^{-2\pi t\cos\alpha}$ when $t \to \infty$. Together with

[22] Since $|e^{2\pi t e^{\alpha i}} - 1| \geqq |e^{2\pi t e^{\alpha i}}| - 1 = e^{2\pi t\cos\alpha} - 1 \geqq e^{2\pi t\sin\epsilon} - 1 \geqq$ either of $2\pi t\sin\epsilon$ or $\dfrac{(2\pi t\sin\epsilon)^3}{3!}$.

(102), this shows that the integral in (105) converges uniformly in respect to s in the region previously considered, and we may therefore integrate (105) in respect to x, obtaining

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + c(y) - \frac{1}{\pi}\int_0^\infty \frac{se^{\alpha i}}{t^2 e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt$$

From (102) and the preceding remarks on the logarithm it follows that the last integral approaches zero as $x \to \infty$, and from (76) and (78) $\log \Gamma(s) - (s - \frac{1}{2}) + s \to \log \sqrt{2\pi}$, so that $c(y)$ equals the latter constant and we finally obtain

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi}$$

(107)

$$- \frac{1}{\pi}\int_0^\infty \frac{se^{\alpha i}}{t^2 e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt,$$

$$|\theta| < \pi, \qquad |\alpha| < \frac{\pi}{2}, \qquad |\theta - \alpha| < \frac{\pi}{2},$$

where, by (76), the last integral equals $\omega(s, 0)$.[23]

Making $s = 1$ and $\alpha = 0$ in (104) and (105), we obtain two expressions[24] for Euler's constant

$$C = \tfrac{1}{2} + 2\int_0^\infty \frac{1}{t^2 + 1} \frac{tdt}{e^{2\pi t} - 1}$$

(108)

$$= \tfrac{1}{2} - \pi \int_0^\infty \frac{1 - t^2}{(t^2 + 1)^2} \log (1 - e^{-2\pi t})dt.$$

To obtain the asymptotic expansion of $\omega(s, 0)$ from (107), we expand the first factor under the integral sign in negative powers of s:

$$\frac{se^{\alpha i}}{t^2 e^{2\alpha i} + s^2} = \sum_{k=1}^{m-1} \frac{(-1)^{k-1} e^{(4k-3)\alpha i} t^{2k-2}}{s^{2k-1}} + \frac{(-1)^{m-1} e^{(4m-3)\alpha i} t^{2m-2}}{s^{2m-3}(t^2 e^{2\alpha i} + s^2)}$$

and obtain

$$\omega(s, 0) = \sum_{k=1}^{m-1} \frac{(-1)^k}{s^{2k-1}} \cdot \frac{1}{\pi}\int_0^\infty e^{(4k-3)\alpha i} t^{2k-2} \log (1 - e^{-2\pi t e^{\alpha i}})dt + R_m,$$

$$R_m = \frac{(-1)^m}{s^{2m-3}} \cdot \frac{1}{\pi}\int_0^\infty \frac{e^{(4m-3)\alpha i} t^{2m-2}}{t^2 e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt.$$

From (102) and (106) it follows that

$$|R_m| \leqq \frac{1}{|s|^{2m-3}} \cdot \frac{1}{\pi}\int_0^\infty \frac{t^{2m-2}}{|s|^2 \cos(\theta - \alpha)} \cdot -\log (1 - e^{-2\pi t \cos \alpha})dt,$$

[23] For $\alpha = 0$, $\Re(s) > 0$, (107) is due to Schaar, Mémoire sur les intégrales eulériennes et sur la convergence d'une certaine classe de séries, Mém. Ac. Belgique, vol. 22 (1848), pp. 3–25.

[24] Poisson, S. D., Mémoire sur les intégrales définies, Journ. de l'École Polytechn., cahier 18 (1813), p. 305.

or introducing $t \cos \alpha$ as integration variable

$$| R_m | \leqq \frac{1}{| s |^{2m-1} \cos^{2m-1} \alpha \cos (\theta - \alpha)} \cdot \frac{1}{\pi} \int_0^\infty - t^{2m-2} \log (1 - e^{-2\pi t}) dt.$$

Hence $| s |^{2m-2} | R_m | \to 0$ as $| s | \to \infty$ and the expansion above is asymptotic, so that it must coincide with (79) for $a = 0$. Expressing $P_{2k}(0)$ in terms of B_k and identifying coefficients, we therefore obtain

$$(109) \qquad \frac{B_k}{(2k - 1) \cdot 2k} = - \frac{1}{\pi} \int_0^\infty e^{(4k-3)\alpha i} t^{2k-2} \log (1 - e^{-2\pi t e^{\alpha i}}) dt;$$

making $k = m$, $\alpha = 0$ and introducing in the last expression for $| R_m |$, we find

$$(110) \qquad | R_m | \leqq \frac{B_m}{(2m - 1) \cdot 2m} \cdot \frac{1}{\cos^{2m-1} \alpha \cos (\theta - \alpha)} \cdot \frac{1}{| s |^{2m-1}}.$$

By choosing α for a given θ so as to make the second denominator a maximum, this form of the remainder will frequently be much smaller than the one in (79).[25]

16. **Sonin's form of the remainder in Stirling's formula.** In the remainder term in $\omega(s, 0)$ obtained at the end of the preceding paragraph, we now assume s real and positive and make $\alpha = 0$, whence

$$(-1)^{m-1} R_m = \frac{1}{s^{2m-3}} \cdot \frac{1}{\pi} \int_0^\infty \frac{t^{2m-2}}{t^2 + s^2} \log \frac{1}{1 - e^{-2\pi t}} dt,$$

or writing $t = su$,

$$(-1)^{m-1} R_m = \frac{1}{\pi} \int_0^\infty u^{2m-2} \cdot \log \frac{1}{1 - e^{-2\pi s u}} \cdot \frac{du}{1 + u^2}.$$

For $s > 0$, $u > 0$, $0 < e^{-2\pi s u} < 1$, and

$$\log \frac{1}{1 - e^{-2\pi s u}} = \sum_1^\infty \frac{e^{-2n\pi s u}}{n}$$
$$= \sum_1^\infty \frac{e^{-2n\pi(s+\lambda)u}}{n} \cdot e^{2n\pi\lambda u};$$

assuming $\lambda > 0$, $e^{2n\pi\lambda u}$ is positive and increases with n, whence

$$\log \frac{1}{1 - e^{-2\pi s u}} > e^{2\pi\lambda u} \cdot \sum_1^\infty \frac{e^{-2n\pi(s+\lambda)u}}{n}$$
$$= e^{2\pi\lambda u} \log \frac{1}{1 - e^{-2\pi(s+\lambda)u}}.$$

Now assume λ so large that for every $u \geqq 0$

[25] For further details on this point, compare Lindelöf, Calcul des résidus, pp. 98–102.

$$e^{2\pi\lambda u} \geqq 1 + u^2, \tag{111}$$

then the last expression for R_m gives

$$\frac{1}{\pi}\int_0^\infty u^{2m-2}\log\frac{1}{1-e^{-2\pi su}}\,du > (-1)^{m-1}R_m$$
$$> \frac{1}{\pi}\int_0^\infty u^{2m-2}\log\frac{1}{1-e^{-2\pi(s+\lambda)u}}\,du,$$

and introducing the new integration variables su in the first and $(s+\lambda)u$ in the second of these integrals

$$\frac{1}{s^{2m-1}}\cdot\frac{1}{\pi}\int_0^\infty t^{2m-2}\log\frac{1}{1-e^{-2\pi t}}\,dt > (-1)^{m-1}R_m$$
$$> \frac{1}{(s+\lambda)^{2m-1}}\cdot\frac{1}{\pi}\int_0^\infty t^{2m-2}\log\frac{1}{1-e^{-2\pi t}}\,dt.$$

Making $k = m$ and $\alpha = 0$ in (109), it follows that

$$\frac{B_m}{(2m-1)\cdot 2m}\cdot\frac{1}{s^{2m-1}} > (-1)^{m-1}R_m > \frac{B_m}{(2m-1)\cdot 2m}\cdot\frac{1}{(s+\lambda)2^{m-1}}.$$

To secure as close a limitation of R_m as possible, we have to find the smallest value of λ which will satisfy (111). It is clear that for this smallest value of λ, the equality sign in (111) must obtain for one or more values of u besides $u = 0$, and consequently the smallest admissible value of λ equals the maximum of

$$\lambda = \frac{1}{2\pi}\cdot\frac{1}{u}\log(1+u^2)$$

This maximum must satisfy the equation

$$\frac{d(2\pi\lambda)}{du} = -\frac{1}{u^2}\log(1+u^2) + \frac{2}{1+u^2} = 0;$$

let

$$f(u^2) = u^2\frac{d(2\pi\lambda)}{du} = \frac{2u^2}{1+u^2} - \log(1+u^2),$$

so that

$$f'(u^2) = \frac{1-u^2}{(1+u^2)^2},$$

and consequently $f(u^2)$ increases for $u < 1$ and decreases for $u > 1$. But for u small, $f(u^2) = u^2 + \cdots > 0$, so that finally $f(u^2) = 0$ can have only one root, for which $u > 1$. The numerical calculation of this root gives $1 + u^2 = 4.92154\cdots$ and the corresponding value of λ is $0.12808\cdots$, so that we may now replace (80) by the following:

$$(112)\quad \omega(s,0) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}B_k}{(2k-1)\cdot 2k} \cdot \frac{1}{s^{2k-1}} + \frac{(-1)^{m-1}B_m}{(2m-1)\cdot 2m} \frac{1}{(s+h)^{2m-1}},$$

$$0 < h < 0.12808\cdots, \quad s > 0.$$

17. Kummer's and Lerch's trigonometric series. The function $\dfrac{\Gamma(s)}{\Gamma(1-s)}$ is holomorphic for $0 < s < 1$, and becomes infinite as $\log 1/s$ when $s \to 0$ and as log $(1-s)$ as $s \to 1$, so that $\displaystyle\int_0^1 \left|\log \frac{\Gamma(s)}{\Gamma(1-s)}\right| ds$ exists. Consequently, this function is expansible in a Fourier series (35) uniformly convergent for $\epsilon \leqq s \leqq 1-\epsilon$, where the coefficients are given by (34):

$$a_n = 2\int_0^1 \log \frac{\Gamma(s)}{\Gamma(1-s)} \cdot \cos 2n\pi s ds, \qquad b_n = 2\int_0^1 \log \frac{\Gamma(s)}{\Gamma(1-s)} \sin 2n\pi s ds,$$

and replacing s by $1-s$ in the expression for a_n, it is seen that

$$a_n = 2\int_0^1 \log \frac{\Gamma(1-s)}{\Gamma(s)} \cos 2n\pi s ds = -a_n$$

or $a_n = 0$. To calculate b_n, we observe that, by $(J\ 3')$,

$$\frac{\Gamma(s)}{\Gamma(1-s)} = \lim_{m\to\infty} m^{2s-1} \cdot \frac{(1-s)(2-s)\cdots(m-s)}{s(1+s)\cdots(m-1+s)},$$

so that

$$\tfrac{1}{2}b_n = \lim_{m\to\infty}\left[\log m \cdot \int_0^1 (2s-1)\sin 2n\pi s ds + \sum_{\nu=1}^{m}\int_0^1 \log(\nu - s)\sin 2n\pi s ds \right.$$

$$\left. + \sum_{\nu=0}^{m-1}\int_0^1 \log(\nu+s)\sin 2n\pi s ds\right].$$

Integrating by parts,

$$\int_0^1 (2s-1)\sin 2n\pi s ds = -\frac{1}{n\pi},$$

and taking $\nu - s$ and $\nu + s$ respectively as new integration variables

$$\sum_{\nu=1}^{m}\int_0^1 \log(\nu-s)\sin 2n\pi s ds = -\sum_{\nu=1}^{m}\int_{\nu-1}^{\nu} \log s \sin 2n\pi s ds$$

$$= -\int_0^m \log s \sin 2n\pi s ds,$$

$$\sum_{\nu=0}^{m-1}\int_0^1 \log(\nu+s)\sin 2n\pi s ds = \sum_{\nu=0}^{m-1}\int_{\nu}^{\nu+1} \log s \sin 2n\pi s ds$$

$$= \int_0^m \log s \sin 2n\pi s ds,$$

whence

$$\tfrac{1}{2}b_n = \lim_{m\to\infty}\left[-\frac{\log m}{n\pi} - 2\int_0^m \log s \sin 2n\pi s ds\right],$$

and integrating the last integral by parts

$$\tfrac{1}{2}b_n = \frac{1}{n\pi}\lim_{m\to\infty}\left[\int_0^m \frac{1-\cos 2n\pi s}{s}ds - \log m\right]$$
$$= \frac{1}{n\pi}\lim_{m\to\infty}\left[\int_0^{1/2}\frac{1-\cos 4mn\pi s}{s}ds - \log mn + \log n\right],$$

so that finally, replacing mn by m in the limit,

$$\tfrac{1}{2}b_n = \frac{1}{n\pi}(\log n + K),$$

where

$$K = \lim_{m\to\infty}\left[\int_0^{1/2}\frac{1-\cos 4m\pi s}{s}ds - \log m\right].$$

To calculate K, we shall replace the integral by one which may be evaluated in finite terms. Observing that

$$\varphi(s) = \frac{1}{s} - \frac{\pi}{\sin \pi s}$$

is holomorphic for $0 \leqq s \leqq \frac{1}{2}$, we may write

$$\int_0^{1/2}\frac{1-\cos 4m\pi s}{s}ds - \pi\int_0^{1/2}\frac{1-\cos 4m\pi s}{\sin \pi s}ds$$
$$= \int_0^{1/2}\varphi(s)ds - \int_0^{1/2}\varphi(s)\cos 4m\pi s ds.$$

Now

$$\int_0^{1/2}\varphi(s)ds = \left[\log\frac{s}{\tan\frac{\pi}{2}s}\right]_{s=0}^{s=1/2} = \log\tfrac{1}{2} + \log\frac{\pi}{2} = \log 2\pi - 3\log 2,$$

and integrating by parts

$$\int_0^{1/2}\varphi(s)\cos 4m\pi s ds = \frac{1}{4m\pi}\Big[\varphi(s)\sin 4m\pi s\Big]_{s=0}^{s=1/2}$$
$$- \frac{1}{4m\pi}\int_0^{1/2}\varphi'(s)\sin 4m\pi s ds$$
$$\to 0 \text{ as } m \to \infty,$$

since $\varphi(s)$ and $\varphi'(s)$ are holomorphic, and therefore bounded, in the interval of integration. Consequently

$$K = \lim_{m\to\infty}\left[\pi\int_0^{1/2}\frac{1-\cos 4m\pi s}{\sin \pi s}\,ds - \log m\right] + \log 2\pi - 3\log 2,$$

and since we have

$$\frac{1-\cos 4m\pi s}{\sin \pi s} = 2\sum_{\nu=1}^{2m}\sin(2\nu-1)\pi s,$$

the integral is readily evaluated, giving

$$K = \lim_{m\to\infty}\left[\sum_{\nu=1}^{2m}\frac{2}{2\nu-1} - \log m\right] + \log 2\pi - 3\log 2.$$

By the definition of Euler's constant,

$$\sum_{\nu=1}^{4m}\frac{2}{\nu} - 2\log 4m \to 2C, \qquad \sum_{\nu=1}^{2m}\frac{1}{\nu} - \log 2m \to C$$

as $m \to \infty$, and by subtraction,

$$\sum_{\nu=1}^{2m}\frac{2}{2\nu-1} - \log m - 3\log 2 \to C,$$

so that finally

$$K = C + \log 2\pi,$$

and consequently

$$\tfrac{1}{2}\log\frac{\Gamma(s)}{\Gamma(1-s)} = \sum_{1}^{\infty}\frac{\log n + C + \log 2\pi}{n\pi}\sin 2n\pi s, \qquad 0 < s < 1.$$

But by (40) we have, for $0 < s < 1$

$$\sum_{1}^{\infty}\frac{C+\log 2\pi}{n\pi}\sin 2n\pi s = (C+\log 2\pi)(\tfrac{1}{2}-s),$$

and furthermore, by (J 6),

$$\tfrac{1}{2}\log\frac{\Gamma(s)}{\Gamma(1-s)} = \tfrac{1}{2}\log\frac{\Gamma(s)^2}{\Gamma(s)\Gamma(1-s)} = \log\Gamma(s) + \tfrac{1}{2}\log\frac{\sin \pi s}{\pi},$$

whence we obtain Kummer's series[26]

$$\log\Gamma(s) + \tfrac{1}{2}\log\frac{\sin\pi s}{\pi} + (C+\log 2\pi)(s-\tfrac{1}{2}) = \sum_{1}^{\infty}\frac{\log n}{n\pi}\sin 2n\pi s, \tag{113}$$

$$0 < s < 1,$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1-\epsilon$.

The differentiation term by term of the series (113) gives a divergent

[26] Kummer, E. E., Beitrag zur Theorie der Function $\Gamma(x)$, Journ. f. Math., vol. 35 (1847), pp. 1–4.

series; a trigonometric expansion of $\psi(s)$ is however furnished by the following theorem:[27]

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{c_n}{n} \sin 2n\pi x$$

be convergent for $0 < x < 1$; then the derivative of $f(x)$ is given in this interval by

$$f'(x) \cdot \frac{\sin \pi x}{\pi} = \sum_{n=0}^{\infty} (c_n - c_{n+1}) \sin (2n + 1)\pi x,$$

where $c_0 = 0$, provided that the last series converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, where ϵ is as small as we please.

Writing

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} (c_n - c_{n+1}) \sin (2n + 1)\pi x \\ &= \sum_{n=0}^{N} (c_n - c_{n+1}) \sin (2n + 1)\pi x + R_N, \\ &= \sum_{n=1}^{N} c_n \sin (2n + 1)\pi x - \sum_{n=1}^{N+1} c_n \sin (2n - 1)\pi x + R_N \\ &= \sum_{n=1}^{N} c_n \cdot 2 \sin \pi x \cos 2n\pi x - c_{N+1} \sin (2N + 1)\pi x + R_N, \end{aligned}$$

where, on account of the uniform convergence, $|R_N|$ may be made as small as we please for $\epsilon \leqq x \leqq 1 - \epsilon$ by taking N sufficiently large, we have

$$\frac{\pi}{\sin \pi x} g(x) = \sum_{n=1}^{N} c_n \cdot 2\pi \cos 2n\pi x - \pi c_{N+1} \frac{\sin (2N + 1)\pi x}{\sin \pi x} + \frac{\pi R_N}{\sin \pi x},$$

and integrating between the limits ϵ and x

$$\int_{\epsilon}^{x} \frac{\pi}{\sin \pi x} g(x) dx = \sum_{n=1}^{N} \frac{c_n}{n} (\sin 2n\pi x - \sin 2n\pi\epsilon) - c_{N+1} \int_{\epsilon}^{x} \frac{\pi \sin (2N + 1)\pi x}{\sin \pi x} dx + \int_{\epsilon}^{x} \frac{\pi R_N}{\sin \pi x} dx.$$

Integrating by parts,

$$c_{N+1} \int_{\epsilon}^{x} \frac{\pi \sin (2N + 1)\pi x}{\sin \pi x} dx = \frac{c_{N+1}}{2N + 1} \left[\frac{\cos (2N + 1)\pi\epsilon}{\sin \pi\epsilon} - \frac{\cos (2N + 1)\pi x}{\sin \pi x} - \int_{\epsilon}^{x} \frac{\pi \cos (2N + 1)\pi x \cos \pi x}{\sin^2 \pi x} dx \right]$$

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[27] Lerch, M., Sur la différentiation d'une classe de séries trigonométriques, Ann. Ec. Norm., ser. 3, vol. 12 (1895), pp. 351–361.

and here the expression in brackets is evidently bounded for $\epsilon \leqq x \leqq 1 - \epsilon$ and all N, so that our integral vanishes as $N \to \infty$, since then $\frac{c_{N+1}}{2N+1} \to 0$[28].

Furthermore,

$$\left| \int_\epsilon^x \frac{\pi R_N}{\sin \pi x} dx \right| < \frac{\pi}{\sin \pi\epsilon} \int_\epsilon^x | R_N | dx \to 0 \text{ as } N \to \infty$$

on account of the uniform convergence of the series $g(x)$, and we therefore have, letting $N \to \infty$,

$$\int_\epsilon^x \frac{\pi}{\sin \pi x} g(x) dx = \sum_{n=1}^{\infty} \frac{c_n}{n} (\sin 2n\pi x - \sin 2n\pi\epsilon) = f(x) - f(\epsilon),$$

whence our proposition follows by differentiation.

In the special case $c_n = \log n/\pi$,

$$g(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \log \frac{n}{n+1} \cdot \sin (2n+1)\pi x,$$

and since $\log \frac{n+1}{n}$ is positive and decreases as n increases, we have by (38)

$$\left| \sum_{\nu=n}^{n+p} \log \frac{\nu+1}{\nu} \sin (2\nu+1)\pi x \right| \leqq \frac{1}{\sin \pi\epsilon} \cdot \log \frac{n+1}{n},$$

so that the series $g(x)$ converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$. We may therefore apply Lerch's theorem to (113) and obtain Lerch's series (l. c.)

$$\begin{aligned} \psi(s) \sin \pi s + \frac{\pi}{2} \cos \pi s + (C + \log 2\pi) \sin \pi s \\ = \sum_{n=1}^{\infty} \log \frac{n}{n+1} \cdot \sin (2n+1)\pi s, \qquad 0 < s < 1, \end{aligned} \tag{114}$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1 - \epsilon$. It should be noted that this series is not a Fourier series according to the definition in § 7.

CHAPTER III.

The Gamma function defined as a definite integral.

18. The Euler integrals of the first and second kind and their elementary properties. In the preceding chapter, the various integrals connected

[28] This is an immediate consequence of the convergence of the series for $f(x)$ in (0, 1) and the following theorem, due to Cantor: When $\sum_0^\infty (a_n \cos nx + b_n \sin nx)$ converges in an interval (a, b), then $a_n \to 0$, $b_n \to 0$ as $n \to \infty$. We omit the proof, since we shall apply Lerch's theorem only to the case $c_n = \log n/\pi$, where it is evident that $c_{n+1}/(2n+1) \to 0$ as $n \to \infty$.

with the Gamma function were derived from the definition of $\Gamma(s)$ as Euler's infinite product. As we shall now proceed to show, the same results may also be obtained by defining $\Gamma(s)$ by means of Euler's integral of the second kind (61)

$$(115) \qquad \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0.$$

As shown in § 10, the integral converges uniformly for $\epsilon \leqq \Re(s) \leqq A$, where ϵ is as small and A as large as we please, but does not converge for $\Re(s) < 0$. Since therefore *s must be restricted to the half-plane* $\Re(s) > 0$ *throughout the present chapter*, certain formulas (for instance, the asymptotic expansions in § 12, which were there proved for all s not on the negative real axis) will have a larger region of validity than that shown by the proofs to be given in the following. Integrating by parts, we find

$$\int_0^\infty e^{-t}t^s dt = \Big[-e^{-t}t^s\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-t}t^{s-1}dt = s\int_0^\infty e^{-t}t^{s-1}dt,$$

or

$$(116) \qquad \Gamma(s+1) = s\Gamma(s),$$

which is one of the fundamental properties of the Gamma function ($J4a$). By direct integration, (61) gives $\Gamma(1) = 1$, whence from (116), for a positive integral n, $\Gamma(n) = (n-1)!$ The formulas (62) and (63)

$$(117) \qquad \frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at}t^{s-1}dt, \qquad a > 0,$$

$$(118) \qquad \frac{d\Gamma(s)}{ds} = \int_0^\infty e^{-t}t^{s-1}\log t\,dt$$

are derived from (115) exactly as in § 10.

Euler's integral of the first kind is defined as

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \qquad \Re(\alpha) > 0, \qquad \Re(\beta) > 0;$$

introducing the new integration variable $u = \dfrac{t}{1-t}$, whence $t = \dfrac{u}{1+u}$, we find

$$(119) \qquad \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\,du.$$

To express this integral in terms of the Gamma function we write $a = 1 + u$, $s = \alpha + \beta$ in (117):

$$\int_0^\infty e^{-(1+u)t}t^{\alpha+\beta-1}dt = \frac{\Gamma(\alpha+\beta)}{(1+u)^{\alpha+\beta}},$$

multiply by $u^{\alpha-1}du$ and integrate from 0 to ∞, which gives

$$\int_0^\infty du \int_0^\infty e^{-(1+u)t}u^{\alpha-1}t^{\alpha+\beta-1}dt = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du.$$

If we replace the integrand to the left by its absolute value (which amounts to replacing α and β by their real parts), the new repeated integral converges, since it equals the right hand member with $\Re(\alpha)$ and $\Re(\beta)$ substituted for α and β; by theorem XXII, we may therefore reverse the order of integration in the repeated integral and obtain

$$\int_0^\infty dt \int_0^\infty e^{-(1+u)t}u^{\alpha-1}t^{\alpha+\beta-1}\,du = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du.$$

But by (117)

$$\int_0^\infty e^{-ut}u^{\alpha-1}du = \frac{\Gamma(\alpha)}{t^\alpha},$$

and consequently

$$\Gamma(\alpha)\int_0^\infty e^{-t}t^{\beta-1}dt = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du,$$

or finally, using (115)

$$(120)\qquad \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\qquad \Re(\alpha) > 0,\qquad \Re(\beta) > 0,$$

which by (119), is equivalent to (71).

To prove that

$$(121)\qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},\qquad 0 < \Re(s) < 1,$$

(which is J 6 with a restriction on $\Re(s)$ imposed by the method of the present chapter), we make $\alpha = s$, $\beta = 1 - s$ in (117), so that

$$\Gamma(\alpha+\beta) = \Gamma(1) = 1,$$

and show that

$$(122)\qquad \int_0^\infty \frac{u^{s-1}}{1+u}du = \frac{\pi}{\sin \pi s},\qquad 0 < \Re(s) < 1.$$

The integral to the left converges uniformly for $\epsilon \leqq \Re(s) \leqq 1 - \epsilon$ by theorem XIX, since $|u^{s-1}/(1+u)| \leqq u^{\epsilon-1}$ for $0 < u < 1$ and $|u^{s-1}/(1+u)| \leqq u^{-\epsilon-1}$ for $u > 1$, and the integrals

$$\int_0^1 u^{s-1}du, \qquad \int_1^\infty u^{-s-1}du$$

both converge. By theorem XXIII, our integral therefore defines a function of s which is holomorphic for $0 < \Re(s) < 1$, and the expression to the right in (121) is also holomorphic in the same region. To show that both sides in (121) are equal throughout this region, it is consequently sufficient to prove their equality for an infinity of values of s having at least one limiting value interior to the region, for instance the values $s = (2p + 1)/2q$, where p and q are any positive integers such that $p < q$. Making $u = t^{2q}$ in (119), it is therefore sufficient to show that

$$\int_{-\infty}^{\infty} \frac{t^{2p}}{1 + t^{2q}}\, dt = 2\int_0^\infty \frac{t^{2p}}{1 + t^{2q}}\, dt = \frac{\pi}{q \sin \dfrac{2p + 1}{2q}\pi}.$$

The roots of $1 + t^{2q} = 0$ are $t_n = e^{(2n+1)\pi i/2q}$ and $\bar{t}_n = e^{-(2n+1)\pi i/2q}$ ($n = 0, 1, \cdots, q - 1$), and decomposing into partial fractions, we have

$$\frac{t^{2p}}{1 + t^{2q}} = \sum_{n=0}^{q-1} \left(\frac{A_n}{t - t_n} + \frac{\bar{A}_n}{t - \bar{t}_n} \right),$$

where

$$A_n = \frac{t_n^{2p}}{2qt_n^{2q-1}} = -\frac{1}{2q} t_n^{2p+1}, \qquad \bar{A}_n = -\frac{1}{2q} \bar{t}_n^{2p+1}.$$

Now $t_n \bar{t}_n = 1$,

$$t_n + \bar{t}_n = 2 \cos \frac{2n + 1}{2q}\pi$$

is real, and

$$\frac{t_n - \bar{t}_n}{2i} = \sin \frac{2n + 1}{2q}\pi > 0;$$

hence

$$\frac{A_n}{t - t_n} + \frac{\bar{A}_n}{t - \bar{t}_n} = \frac{(A_n + \bar{A}_n)t - (A_n \bar{t}_n + \bar{A}_n t_n)}{t^2 - (t_n + \bar{t}_n)t + 1}$$

$$= \frac{1}{2} \frac{(A_n + \bar{A}_n)(2t - t_n - \bar{t}_n)}{t^2 - (t_n + \bar{t}_n)t + 1}$$

$$+ \frac{1}{2} \frac{(A_n - \bar{A}_n)(t_n - \bar{t}_n)}{\left(t - \dfrac{t_n + \bar{t}_n}{2}\right)^2 + \left(\dfrac{t_n - \bar{t}_n}{2i}\right)^2}$$

and

$$\lim_{t\to\infty}\int_{-t}^{t}\left(\frac{A_n}{t-t_n}+\frac{\bar{A}_n}{t-\bar{t}_n}\right)dt = \tfrac{1}{2}(A_n+\bar{A}_n)\lim_{t\to\infty}\log\frac{t^2-(t_n+\bar{t}_n)t+1}{t^2+(t_n+\bar{t}_n)t+1}$$

$$+\tfrac{1}{2}(A_n-\bar{A}_n)(t_n-\bar{t}_n)\cdot\frac{1}{\dfrac{t_n-\bar{t}_n}{2i}}$$

$$\times\left[\arctan\frac{t-\dfrac{t_n+\bar{t}_n}{2}}{\dfrac{t_n-\bar{t}_n}{2i}}\right]_{t=-\infty}^{t=\infty}$$

$$=\pi i(A_n-\bar{A}_n) = 2\pi\Re(iA_n)$$

$$=\frac{\pi}{q}\Re(-ie^{(2n+1)(2p+1)\pi i/2q}),$$

so that finally

$$\int_{-\infty}^{\infty}\frac{t^{2p}}{1+t^{2q}}dt = \frac{\pi}{q}\Re\left(\sum_{n=0}^{q-1}-ie^{(2n+1)(2p+1)\pi i/2q}\right)$$

$$=\frac{\pi}{q}\Re\left(-ie^{(2p+1)\pi i/2q}\cdot\frac{e^{2q(2p+1)\pi i/2q}-1}{e^{2(2p+1)\pi i/2q}-1}\right)$$

$$=\frac{\pi}{q}\Re\left(-i\cdot\frac{-1-1}{e^{(2p+1)\pi i/2q}-e^{-(2p+1)\pi i/2q}}\right)$$

$$=\frac{\pi}{q}\cdot\frac{1}{\sin\dfrac{2p+1}{2q}\pi},$$

which proves our proposition.[29]

[29] From (121) we may obtain the expansion in partial fractions of $\pi/\sin\pi s$. For decomposing the integral in two with limits 0, 1 and 1, ∞, and changing u into $1/u$ in the second of these integrals, we obtain

$$\frac{\pi}{\sin\pi s} = \int_0^1\frac{u^{s-1}+u^{-s}}{1+u}du, \qquad 0<\Re(s)<1,$$

and since

$$\frac{1}{1+u} = \sum_{\nu=0}^{n-1}(-1)^\nu u^\nu + (-1)^n\frac{u^n}{1+u},$$

it is seen that

$$\frac{\pi}{\sin\pi s} = \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{s+\nu}+\sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{\nu+1-s}+(-1)^n\int_0^1\frac{u^{s+n-1}+u^{n-s}}{1+u}du.$$

For $\epsilon \leqq \Re(s) \leqq 1-\epsilon$, we have

$$\left|\int_0^1\frac{u^{s+n-1}+u^{n-s}}{1+u}du\right| < \int_0^1(u^{\epsilon+n-1}+u^{n-(1-\epsilon)})ds = \frac{2}{n+\epsilon}\to 0 \text{ as } n\to\infty,$$

and consequently

$$\frac{\pi}{\sin\pi s} = \sum_0^\infty\frac{(-1)^\nu}{s+\nu}+\sum_0^\infty\frac{(-1)^\nu}{\nu+1-s} = \sum_{-\infty}^{\infty}\frac{(-1)^\nu}{s+\nu},$$

In (120), we now make $\alpha = \beta = s$, obtaining

$$\frac{\Gamma(s)^2}{\Gamma(2s)} = \int_0^1 \frac{u^{s-1}}{(1+u)^{2s}}du + \int_1^\infty \frac{u^{s-1}}{(1+u)^{2s}}du = 2\int_0^1 \frac{u^{s-1}}{(1+u)^{2s}}du,$$

and writing $u = (1 - t^{1/2})/(1 + t^{1/2})$ in the last integral,

$$\frac{\Gamma(s)^2}{\Gamma(2s)} = \frac{1}{2^{2s-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}dt = \frac{1}{2^{2s-1}}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})}$$

by (119) and (120). Since (121) gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain Legendre's formula (J^{13})

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s), \qquad \Re(s) > 0. \tag{123}$$

19. **Integrals for $\psi(s)$. The infinite product for $\Gamma(s)$.** In (118), we replace $\log t$ by the expression

$$\log t = \int_0^\infty \frac{e^{-u} - e^{-tu}}{u}du$$

obtained from (87), and find

$$\Gamma'(s) = \int_0^\infty dt \int_0^\infty e^{-t}t^{s-1}\frac{e^{-u} - e^{-tu}}{u}du = \int_0^\infty dt\int_0^\infty f(t,u)du;$$

since $e^{-u} \gtrless e^{-tu}$ as $t \gtrless 1$, we have

$$|f(t,u)| = \pm e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u} \qquad (\pm \text{ as } t \gtrless 1),$$

so that

$$\begin{aligned}\int_0^\infty dt\int_0^\infty |f(t,u)|du &= \int_0^1 dt\int_0^\infty |f(t,u)|du + \int_1^\infty dt\int_0^\infty |f(t,u)|du \\ &= -\int_0^1 dt\int_0^\infty e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u}du \\ &\quad + \int_1^\infty dt\int_0^\infty e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u}du\end{aligned}$$

and performing the integrations in respect to u

$$\int_0^\infty dt\int_0^\infty |f(t,u)|du = -\int_0^1 e^{-t}t^{\Re(s)-1}\log t\,dt + \int_1^\infty e^{-t}t^{\Re(s)-1}\log t\,dt.$$

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for $0 < \Re(s) < 1$, but since the infinite series converges uniformly in any finite region of the s-plane to which the points $s = 0, \pm 1, \pm 2, \cdots$ are exterior, and consequently the series is a function of s holomorphic in this region, the above equation is valid for *all* values of s distinct from $0, \pm 1, \pm 2, \cdots$. Conversely, we may deduce this partial fraction expansion from that of the cotangent by means of the identity $2/\sin \pi s = \cot \pi s/2 + \cot [\pi(1-s)]/2$, and then the above expansion of the integral gives us a new proof of (121).

Both of the last integrals converge since $\Re(s) > 0$; hence $\int_0^\infty dt \int_0^\infty |f(t, u)| \, du$ converges and by theorem XXII, we may reverse the order of integration in the repeated integral for $\Gamma'(s)$, whence

$$\Gamma'(s) = \int_0^\infty du \int_0^\infty t^{s-1} \frac{e^{-u-t} - e^{-(1+u)t}}{u} dt$$

$$= \int_0^\infty \left[e^{-u}\Gamma(s) - \frac{\Gamma(s)}{(1+u)^s} \right] \frac{du}{u}$$

by (115) and (117), or finally, writing t instead of u in the integral,

$$(124) \qquad \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty (e^{-t} - (1+t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0,$$

which is identical with (90). We now define Euler's constant by the relation

$$(125) \qquad -C = \psi(1) = \Gamma'(1) = \int_0^\infty (e^{-t} - (1+t)^{-1}) \frac{dt}{t},$$

whence by subtraction

$$\psi(s) + C = \int_0^\infty ((1+t)^{-1} - (1+t)^{-s}) \frac{dt}{t},$$

and replacing $1 + t$ by e^t

$$(126) \qquad \psi(s) + C = \int_0^\infty \frac{e^{-t} - e^{-st}}{1 - e^{-t}} dt.$$

Differentiating under the integral sign,

$$(127) \qquad \psi'(s) = \int_0^\infty \frac{te^{-st}}{1 - e^{-t}} dt,$$

where evidently $\left| \dfrac{1}{1 - e^{-t}} \right| < M$ for $t \geqq 0$, so that for $\Re(s) \geqq \epsilon$

$$\int_0^\infty \left| \frac{te^{-st}}{1 - e^{-t}} \right| dt \leqq \int_0^\infty M te^{-\epsilon t} dt = \frac{M}{\epsilon^2}.$$

Consequently the integral in (127) converges uniformly for $\Re(s) \geqq \epsilon$, so that the differentiation under the integral sign is legitimate by theorem XXI. We have

$$\frac{te^{-st}}{1 - e^{-t}} = \sum_{\nu=0}^\infty te^{-(s+\nu)t}, \qquad \left| \frac{te^{-st}}{1 - e^{-t}} \right| = \frac{te^{-\Re(s) \cdot t}}{1 - e^{-t}} = \sum_{\nu=0}^\infty ta^{-(\Re(s)+\nu)t},$$

and since $\int_0^\infty \left| \dfrac{te^{-st}}{1 - e^{-t}} \right| dt$ converges for $\Re(s) > 0$, theorem XXII cor.

gives

$$\psi'(s) = \sum_{\nu=0}^{\infty} \int_0^{\infty} te^{-(s+\nu)t}dt,$$

or evaluating the integrals by means of (117)

$$\psi'(s) = \sum_{\nu=0}^{\infty} \frac{1}{(s+\nu)^2} = \frac{d}{ds} \sum_{\nu=0}^{\infty} \left(\frac{1}{1+\nu} - \frac{1}{s+\nu} \right)$$

where the last series is uniformly convergent for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, since

$$\left| \frac{1}{1+\nu} - \frac{1}{s+\nu} \right| = \frac{|s-1|}{(1+\nu)\,|s+\nu|} < \frac{A+1}{(1+\nu)(\epsilon+\nu)} < \frac{A+1}{\nu^2}$$

and consequently, remembering that $\psi(1) = -C$,

$$\psi(s) + C = \sum_{\nu=0}^{\infty} \left(\frac{1}{1+\nu} - \frac{1}{s+\nu} \right).$$

This may also be written

$$\frac{d \log \Gamma(s+1)}{ds} + C = \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{s+\nu} \right) = \frac{d}{ds} \sum_{\nu=1}^{\infty} \left(\frac{s}{\nu} - \log \frac{s+\nu}{\nu} \right),$$

the last series being uniformly convergent as before, as is readily seen by expanding the logarithm, and since for $s = 0$, $\Gamma(s+1) = 1$, it follows that

$$\log \Gamma(s+1) + Cs = \sum_{\nu=1}^{\infty} \left(\frac{s}{\nu} - \log \left(1 + \frac{s}{\nu} \right) \right).$$

For $s = 1$, we obtain, since $\Gamma(2) = 1$,

$$C = \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \log \left(1 + \frac{1}{\nu} \right) \right)$$

$$= \lim_{n \to \infty} \left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n-1} - \log n \right),$$

which is the definition of Euler's constant given in J § 1, and passing from logarithms to numbers, the preceding equation gives, since $\Gamma(s+1) = s\Gamma(s)$,

$$\frac{1}{\Gamma(s)} = e^{Cs} \cdot s \prod_{\nu=1}^{\infty} \left(1 + \frac{s}{\nu} \right) e^{-s/\nu}, \qquad \Re(s) > 0.$$

Except for the condition $\Re(s) > 0$, this is the product formula (J 5).

20. **Integrals for** $\log \Gamma(s)$. **Raabe's integral.** In the same way as in the case of (89), we may show that the integral (126) converges uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, and consequently

$$(s-1)\psi(1+u(s-1))+C(s-1)=\int_0^\infty (s-1)\frac{e^{-t}-e^{-t-u(s-1)t}}{1-e^{-t}}dt$$

converges uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, $0 \leqq u \leqq 1$. By theorem XX, the integration in respect to u between the limits 0 and 1 may therefore be performed under the integral sign to the right and gives, since $\Gamma(1) = 1$,

$$\log \Gamma(s) + C(s-1) = \int_0^\infty \frac{(s-1)te^{-t}+e^{-st}-e^{-t}}{1-e^{-t}}\frac{dt}{t}.$$

For $s = 2$ we obtain, since $\Gamma(2) = 1$,

$$C = \int_0^\infty \left(\frac{te^{-t}}{1-e^{-t}} - e^{-t}\right)\frac{dt}{t},$$

and multiplying by $(s - 1)$ and subtracting from the preceding equation,

$$(128) \quad \log \Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st}-e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}, \qquad \Re(s) > 0,$$

which is identical with (94) and is seen, as in the case of (89), to converge uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$. We shall now evaluate Raabe's integral

$$\int_0^1 \log \Gamma(s+a)da, \qquad \Re(s) > 0,$$

where, on account of the uniform convergence of (128), $\log \Gamma(s + a)$ is continuous for $\Re(s)>0$ and $0 \leqq a \leqq 1$. From $\log \Gamma(a) = \log \Gamma(1 + a) - \log a$ and the existence of $\int_0^1 \log a da$ it therefore follows that $\int_0^1 \log \Gamma(a)da$ exists. The integral

$$I = \int_0^1 \log \Gamma(us+a)da,$$

where $\Re(s) \geqq \epsilon$, $|s| \leqq A$, $\delta \leqq u \leqq 1$ (δ being arbitrarily small) may be differentiated under the integral sign, since the resulting integral

$$\frac{\partial I}{\partial u} = \int_0^1 s\psi(us+a)da$$

converges uniformly for the values of s and u considered. But integrating in respect to a and applying (116), we find

$$\frac{\partial I}{\partial u} = s(\log \Gamma(us+1) - \log \Gamma(us)) = s \log us,$$

whence integrating in respect to u between the limits 0 and 1,

$$\int_0^1 \log \Gamma(s+a)da - \int_0^1 \log \Gamma(a)da = \int_0^1 s \log usdu = s \log s - s.$$

Now, replacing a by $1 - a$,

$$\int_0^1 \log \Gamma(a)da = \int_0^1 \log \Gamma(1-a)da$$

and consequently, applying (121)

$$\int_0^1 \log \Gamma(a)da = \tfrac{1}{2}\int_0^1 (\log \Gamma(a) + \log \Gamma(1-a))da$$

$$= \tfrac{1}{2}\int_0^1 (\log \pi - \log \sin \pi a)da = \log \sqrt{\pi} - \tfrac{1}{2}\int_0^1 \log \sin \pi a da.$$

To the last integral, we apply the formula $\sin \pi a = 2 \sin \frac{\pi a}{2} \sin \frac{\pi(1+a)}{2}$, whence

$$\int_0^1 \log \sin \pi a da = \log 2 + \int_0^1 \log \sin \frac{\pi a}{2} da + \int_0^1 \log \sin \frac{\pi(1+a)}{2} da$$

$$= \log 2 + 2\int_0^{\frac{1}{2}} \log \sin \pi a da + 2\int_0^{\frac{1}{2}} \log \sin \left(\frac{\pi}{2} + \pi a\right) da$$

$$= \log 2 + 2\int_0^{\frac{1}{2}} \log \sin \pi a da + 2\int_{\frac{1}{2}}^1 \log \sin \pi a da$$

$$= \log 2 + 2\int_0^1 \log \sin \pi a da,$$

so that

$$\int_0^1 \log \sin \pi a da = -\log 2$$

and

$$\int_0^1 \log \Gamma(a)da = \log \sqrt{\pi} + \tfrac{1}{2}\log 2 = \log \sqrt{2\pi},$$

and finally

$$\int_0^1 \log \Gamma(s+a)da = s \log s - s + \log \sqrt{2\pi}, \qquad \Re(s) > 0. \tag{129}$$

We may obtain another expression for Raabe's integral by replacing s by $s + a$ in (128):

$$\log \Gamma(s+a) = \int_0^\infty \left[(s+a-1)e^{-t} + \frac{e^{-st-at} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t}, \tag{130}$$

$$\Re(s) > 0, \qquad 0 \leqq a < 1$$

and integrating in respect to a from 0 to 1; the integral to the right being uniformly convergent in respect to a from $0 \leqq a \leqq 1$, we may perform the integration under the integral sign and obtain

$$(131)\quad \int_0^1 \log \Gamma(s+a)da = \int_0^\infty \left[(s-\tfrac{1}{2})e^{-t} + \frac{e^{-st}}{t} - \frac{e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}, \quad \Re(s)>0.$$

21. Binet's integral and the asymptotic expansion of $\log \Gamma(s+a)$ and $\psi(s+a)$. From (130) we subtract (131) and find by the aid of (129):

$$\log \Gamma(s+a) - (s \log s - s + \log \sqrt{2\pi})$$
$$= \int_0^\infty \left[(a-\tfrac{1}{2})e^{-t} - \frac{e^{-st}}{t} + \frac{e^{-st-at}}{1-e^{-t}}\right]\frac{dt}{t};$$

from (87) we obtain

$$\log s = \int_0^\infty \frac{e^{-t}-e^{-st}}{t}\,dt,$$

and multiplying by $a - \frac{1}{2}$ and subtracting from the preceding equation,

$$\log \Gamma(s+a) = (s+a-\tfrac{1}{2}) \log s - s + \log \sqrt{2\pi}$$
$$+ \int_0^\infty \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\frac{e^{-st}dt}{t}, \qquad \Re(s) > 0,$$

Rewriting equations (97) and (98),

$$(132)\qquad \omega(s,a) = \int_0^\infty \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\frac{e^{-st}dt}{t},$$

$$(133)\qquad f(t,a) = \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\cdot\frac{1}{t},$$

we find as in § 14, upon successive integration by parts in (132),

$$(134)\quad \omega(s,a) = \left[-\sum_{k=1}^{2m-2} \frac{\partial^{k-1} f(t,a)}{\partial t^{k-1}} \frac{e^{-st}}{s^k}\right]_{t=0}^{t=\infty} + \frac{1}{s^{2m-2}} \int_0^\infty \frac{\partial^{2m-2} f(t,a)}{\partial t^{2m-2}} e^{-st}dt.$$

It is now proposed to show that this formula yields the asymptotic expansion (78) for $\Re(s) > 0$, and the proof will be conducted without any comparison of (134) to (77) (since we have not established the latter formula by the methods of the present chapter). To this purpose, we note that $f(t, a)$ is a function of a holomorphic for $0 \leqq a \leqq 1$, and may therefore be expanded in a Fourier series uniformly convergent in this interval

$$f(t,a) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos 2n\pi a + b_n \sin 2n\pi a)$$
$$= \frac{a_0}{2} + \sum_1^\infty (A_n e^{2n\pi ia} + A_{-n} e^{-2n\pi ia}),$$

where, by (34),

$$A_n = \tfrac{1}{2}(a_n - ib_n) = \int_0^1 f(t, a)e^{-2n\pi i a}da, \quad A_{-n} = \tfrac{1}{2}(a_n + ib_n) = \int_0^1 f(t, a)e^{2n\pi i a}da,$$

From (133) we find

$$a_0 = 2\int_0^1 f(t, a)da = 0$$

and

$$A_{-n} = \frac{1}{t}\int_0^1 \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}}\right)e^{2n\pi i a}da$$

$$= \frac{1}{t}\left[\frac{1}{2n\pi i}\,a e^{2n\pi i a}\right]_{a=0}^{a=1} + \frac{1}{t}\left[\frac{1}{2n\pi i - t}\cdot\frac{e^{(2n\pi i - t)a}}{1 - e^{-t}}\right]_{a=0}^{a=1}$$

$$= \frac{1}{t}\left(\frac{1}{2n\pi i} - \frac{1}{2n\pi i - t}\right) = -\frac{1}{2n\pi i(2n\pi i - t)};$$

A_n is obtained from this by changing the sign of i, so that

$$(135) \qquad f(t, a) = -\sum_{n=1}^{\infty}\left(\frac{e^{2n\pi i a}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi i a}}{2n\pi i(2n\pi i - t)}\right).$$

Differentiating term by term in respect to t, which is legitimate since the series obtained are uniformly convergent for $t \geqq 0$, it is seen that

$$(136) \quad \frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}} = -(k-1)!\sum_{n=1}^{\infty}\left(\frac{(-1)^{k-1}e^{2n\pi i a}}{2n\pi i(2n\pi i + t)^k} + \frac{e^{-2n\pi i a}}{2n\pi i(2n\pi i - t)^k}\right)$$

and from the uniform convergence for $t \geqq 0$ it follows that $\dfrac{\partial^{k-1}f(t, a)}{\partial t^{k-1}} \to 0$ as $t \to \infty$. For $t = 0$, we obtain

$$\left[\frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}}\right]_{t=0} = -(k-1)!\sum_{n=1}^{\infty}\frac{(-1)^{k-1}e^{2n\pi i a} + e^{-2n\pi i a}}{(2n\pi i)^{k+1}},$$

and comparing the cases k odd and k even separately to (42), we find

$$\left[\frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}}\right]_{t=0} = (-1)^{k(k-1)/2}(k-1)!P_{k+1}(a).$$

Since $|\,2n\pi i \pm t\,| \geqq 2n\pi$, it follows from (136) for $k = 2m - 1$ that

$$\left|\frac{\partial^{2m-2}f(t, a)}{\partial t^{2m-2}}\right| < (2m-2)!\sum_{1}^{\infty}\frac{2}{(2n\pi)^{2m}} = (2m-2)!P_{2m}(0),$$

and for $s = |\,s\,|\,e^{\theta i}$ $(-\pi/2 < \theta < \pi/2$ since $\Re(s) > 0)$,

$$\left|\int_0^\infty \frac{\partial^{2m-2}f(t,a)}{\partial t^{2m-2}} e^{-st}dt\right| < (2m-2)!P_{2m}(0)\int_0^\infty e^{-|s|\cos\theta\cdot t}dt$$

$$= (2m-2)!\frac{P_{2m}(0)}{|s|\cos\theta}.$$

Introducing all this in (134), we finally obtain

$$\omega(s,a) = \sum_{k=1}^{2m-2}(-1)^{k(k-1)/2}\frac{(k-1)!P_{k+1}(a)}{s^k} + \frac{(2m-2)!P_{2m}(0)\cdot h}{|s|^{2m-1}\cos\theta},$$

(137)

$$|h| < 1, \qquad 0 \leqq a < 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

which, except for the slightly different form of the remainder term, is identical to (78).

We have

$$\omega^*(s,a) = \log s - \psi(s+a) = (\tfrac{1}{2} - a)\frac{1}{s} - \frac{\partial\omega(s,a)}{\partial s},$$

and by theorem XXI, it is legitimate to differentiate in respect to s under the integral sign in (132), whence

(138) $$\omega^*(s,a) = (\tfrac{1}{2} - a)\frac{1}{s} + \int_0^\infty tf(t,a)e^{-st}dt, \qquad \Re(s) > 0.$$

Integrating by parts, and using (135) in the same manner as above, we find

$$\omega^*(s,a) = (\tfrac{1}{2} - a)\frac{1}{s} + \sum_{k=2}^{2m-1}(-1)^{(k-1)(k-2)/2}\frac{(k-1)!P_k(a)}{s^k}$$

(139) $$+ \frac{(2m-1)!P_{2m}(0)\cdot h}{|s|^{2m}\cos\theta},$$

$$|h| < 1, \qquad 0 \leqq a < 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

which differs from (84) only in the form of the remainder term.

22. **The integrals of Schaar and Landsberg.** Let us first assume *s real and positive* and replace $f(t,a)$ in (132) by the series (135), so that

$$\omega(s,a) = -\int_0^\infty \sum_{n=1}^\infty \left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)}\right)e^{-st}dt.$$

Since $|2n\pi i \pm t| \geqq 2n\pi$, the absolute value of the general term in the series does not exceed $\left(\frac{1}{(2n\pi)^2} + \frac{1}{(2n\pi)^2}\right)e^{-st}$, and the integral

$$\int_0^\infty \sum_{n=1}^\infty \frac{1}{2\pi^2 n^2} e^{-st} dt = \frac{1}{2\pi^2 s} \sum_1^\infty \frac{1}{n^2}$$

being convergent, the same is the case with

$$\int_0^\infty \sum_{n=1}^\infty \left| \left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)} \right) e^{-st} \right| dt.$$

By theorem XXII cor., we may therefore reverse the order of integration and summation in the expression for $\omega(s, a)$, and write

$$\omega(s, a) = -\sum_{n=1}^\infty \int_0^\infty \left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)} \right) e^{-st} dt.$$

Replacing t by $2n\pi t/s$ in each integral (this is a real substitution since s is real), we obtain

$$\omega(s, a) = \sum_{n=1}^\infty \int_0^\infty \frac{1}{2n\pi} \left(\frac{e^{-2n\pi ia}}{s + ti} + \frac{e^{2n\pi ia}}{s - ti} \right) e^{-2n\pi t} dt;$$

since $|s \pm ti| \geqq s$, the absolute value of each integrand does not exceed $(1/n\pi s)e^{-2n\pi t}$, and the series

$$\sum_{n=1}^\infty \int_0^\infty \frac{e^{-2n\pi t}}{n\pi s} dt = \frac{1}{2\pi^2 s} \sum_1^\infty \frac{1}{n^2}$$

being convergent, the same is the case with the series

$$\sum_{n=1}^\infty \int_0^\infty \left| \frac{1}{2n\pi} \left(\frac{e^{-2n\pi ia}}{s + ti} + \frac{e^{2n\pi ia}}{s - ti} \right) e^{-2n\pi t} \right| dt.$$

By theorem XXII cor., we may therefore again reverse the order of summation and integration in $\omega(s, a)$, which gives

$$\omega(s, a) = \int_0^\infty \sum_{n=1}^\infty \frac{1}{2n\pi} \left(\frac{e^{-2n\pi(t+ai)}}{s + ti} + \frac{e^{-2n\pi(t-ai)}}{s - ti} \right) dt$$

and summing the infinite series

$$\omega(s, a) = \frac{1}{2\pi} \int_0^\infty \left(\frac{1}{s + ti} \log \frac{1}{1 - e^{-2\pi(t+ai)}} + \frac{1}{s - ti} \log \frac{1}{1 - e^{-2\pi(t-ai)}} \right) dt. \tag{140}$$

Passing to the case of s complex, we may now write for $\Re(s) > 0$

$$\omega(s, a) = \frac{1}{2\pi} \int_0^\infty \frac{1}{s + ti} \log \frac{1}{1 - e^{-2\pi(t+ai)}} dt + \frac{1}{2\pi} \int_0^\infty \frac{1}{s - ti} \log \frac{1}{1 - e^{-2\pi(t-ai)}} dt. \tag{141}$$

In fact, $1/(s + ti)$ and $1/(s - ti)$ are holomorphic in s for $\Re(s) \geqq \epsilon$, $|s| \leqq A$ and $t \geqq 0$; since $|s \pm ti| \geqq \epsilon$ and

$$\left|\log \frac{1}{1 - e^{-2\pi(t\pm ai)}}\right| \leqq \log \frac{1}{1 - e^{-2\pi t}}$$

(which is seen by expansion in series, as in (106)), and furthermore

$$\int_0^\infty \frac{1}{\epsilon} \log \frac{1}{1 - e^{-2\pi t}}\, dt$$

converges (the integrand becoming infinite as $\log t$ for $t \to 0$ and vanishing as $e^{-2\pi t}$ for $t \to \infty$), both integrals in (141) converge uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$. The application of theorem XXIII now shows that the right side in (141) is a function of s holomorphic for $\Re(s) > 0$; the same being the case with $\omega(s, a)$, and both sides in (141) being equal for s real and positive by (140), it follows that the equality subsists in the entire half-plane $\Re(s) > 0$.

Writing $1 - e^{-2\pi(t\pm ai)} = re^{\pm\theta i}$, we have $r^2 = 1 - e^{-2\pi t} \cos 2\pi a + e^{-4\pi t}$,

$$\theta = \arctan \frac{\sin 2\pi a}{e^{2\pi t} - \cos 2\pi a},$$

and making

$$(142)\qquad \begin{aligned} \psi(t, a) &= \frac{1}{4\pi} \log (1 - e^{-2\pi t} \cos 2\pi a + e^{-4\pi t}), \\ \chi(t, a) &= -\frac{1}{2\pi} \arctan \frac{\sin 2\pi a}{e^{2\pi t} - \cos 2\pi a}, \end{aligned}$$

we find by a simple algebraic transformation of (141)

$$(143)\qquad \omega(s, a) = \int_0^\infty \frac{2t\chi(t, a)}{s^2 + t^2}\, dt - \int_0^\infty \frac{2s\psi(t, a)}{s^2 + t^2}\, dt, \qquad \Re(s) > 0.$$

Differentiating (141) in respect to $x = \Re(s)$, we find

$$(144)\qquad \begin{aligned} (\tfrac{1}{2} - a)\frac{1}{s} - \omega^*(s, a) &= \frac{\partial\omega(s, a)}{\partial s} \\ &= -\frac{1}{2\pi}\int_0^\infty \frac{1}{(s + ti)^2} \log \frac{1}{1 - e^{-2\pi(t+ai)}}\, dt \\ &\quad - \frac{1}{2\pi}\int_0^\infty \frac{1}{(s - ti)^2} \log \frac{1}{1 - e^{-2\pi(t-ai)}}\, dt; \end{aligned}$$

the differentiation under the integral sign is legitimate by theorem XXI, since the integrals obtained converge uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, as is seen in exactly the same way as for the corresponding integrals in

(141). Integrating by parts in the formula just obtained, we find for $0 < a < 1$

$$-\frac{1}{2\pi}\int_0^\infty \frac{1}{(s+ti)^2}\log\frac{1}{1-e^{-2\pi(t+ai)}}\,dt = \left[\frac{1}{2\pi i}\cdot\frac{1}{s+ti}\log\frac{1}{1-e^{-2\pi(t+ai)}}\right]_{t=0}^{t=\infty} - i\int_0^\infty \frac{1}{s+ti}\cdot\frac{e^{-2\pi(t+ai)}}{1-e^{-2\pi(t+ai)}}\,dt,$$

and the first term to the right equals

$$\frac{1}{2\pi i s}\log(1-e^{-2\pi a i}) = \frac{1}{2\pi i s}\log\left(2e^{\pi(1/2-a)i}\cdot\frac{e^{\pi a i}-e^{-\pi a i}}{2i}\right) = \frac{1}{2\pi i s}[\pi(\tfrac{1}{2}-a)i + \log(2\sin\pi a)].$$

The second integral in (144) is found by changing the sign of i, and we finally see that

$$\begin{aligned}\omega^*(s,a) &= i\int_0^\infty \frac{1}{s+ti}\cdot\frac{e^{-2\pi(t+ai)}}{1-e^{-2\pi(t+ai)}}\,dt \\ &\quad - i\int_0^\infty \frac{1}{s-ti}\cdot\frac{e^{-2\pi(t-ai)}}{1-e^{-2\pi(t-ai)}}\,dt, \quad \Re(s) > 0, \quad 0 < a < 1,\end{aligned} \tag{145}$$

or writing

$$\begin{aligned}\psi_1(t,a) &= \frac{\cos 2\pi a - e^{-2\pi t}}{e^{2\pi t} - 2\cos 2\pi a + e^{-2\pi t}}, \\ \chi_1(t,a) &= \frac{\sin 2\pi a}{e^{2\pi t} - 2\cos 2\pi a + e^{-2\pi t}},\end{aligned} \tag{146}$$

and transforming (145) algebraically

$$\omega^*(s,a) = \int_0^\infty \frac{2s\chi_1(t,a)}{s^2+t^2}\,dt + \int_0^\infty \frac{2t\psi_1(t,a)}{s^2+t^2}\,dt, \qquad \Re(s) > 0. \tag{147}$$

(147) is also seen to be true for $a = 0$ by making $a = 0$ in (143), differentiating in respect to x and integrating by parts. In the special cases $a = 0$ and $a = \frac{1}{2}$ these expressions are considerably simplified, and we obtain

$$\begin{aligned}\omega(s,0) &= \frac{1}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\log\frac{1}{1-e^{-2\pi t}}\,dt, \\ \omega^*(s,0) &= \int_0^\infty \frac{2t}{s^2+t^2}\frac{dt}{e^{2\pi t}-1}, \\ \omega(s,\tfrac{1}{2}) &= \frac{1}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\log\frac{1}{1+e^{-2\pi t}}\,dt, \\ \omega^*(s,\tfrac{1}{2}) &= \int_0^\infty \frac{2t}{s^2+t^2}\frac{dt}{e^{2\pi t}+1}.\end{aligned} \tag{148}$$

The integrals (148) are due to Schaar,[23] and (143) and (147) to Landsberg.[30] Using the identity

$$\frac{1}{s^2+t^2} = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}t^{2k-2}}{s^{2k}} + \frac{(-1)^{m-1}t^{2m-2}}{s^{2m-2}(s^2+t^2)}$$

in (143) and (147), and proceeding in the same manner as at the end of § 15, there result asymptotic expansions which must coincide with (137) and (139). Comparing coefficients, we readily obtain

$$(149)\qquad \begin{aligned} P_{2k}(a) &= -\frac{2}{(2k-2)!}\int_0^\infty \psi(t, a)t^{2k-2}dt \\ &= \frac{2}{(2k-1)!}\int_0^\infty \psi_1(t, a)t^{2k-1}dt, \\ P_{2k+1}(a) &= -\frac{2}{(2k-1)!}\int_0^\infty \chi(t, a)t^{2k-1}dt \\ &= \frac{2}{(2k)!}\int_0^\infty \chi_1(t, a)t^{2k}dt. \end{aligned}$$

23. **Kummer's trigonometric series.** Let s be real and $\epsilon \leqq s \leqq 1-\epsilon$; we then obtain from (128)

$$\log\frac{\Gamma(s)}{\Gamma(1-s)} = \int_0^\infty \left[\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}} + (2s-1)e^{-t}\right]\frac{dt}{t},$$

and subtracting (125) multiplied by $2s-1$

$$(150)\qquad \log\frac{\Gamma(s)}{\Gamma(1-s)} + C(2s-1) = \int_0^\infty \left[\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}} + \frac{2s-1}{1+t}\right]\frac{dt}{t}.$$

We now have the Fourier expansion

$$\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}} + \frac{2s-1}{1+t} = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n\cos 2n\pi s + b_n \sin 2n\pi s)$$

uniformly convergent for $0 \leqq s \leqq 1$ and a fixed value of t, since the left side is a holomorphic function of s. It is seen from (34) that $a_0 = 0$ and that for $n > 0$

[30] Landsberg, G., Sur un nouveau développement de la fonction gamma, Mém. Ac. Belgique, vol. 55 (1897).

$$a_n + ib_n = 2\int_0^1 \left(\frac{e^{-st} - e^{-(1-s)t}}{1 - e^{-t}} + \frac{2s - 1}{1 + t}\right) e^{2n\pi i s} ds$$

$$= \frac{2}{1 - e^{-t}} \left[\frac{e^{(2n\pi i - t)s}}{2n\pi i - t} - \frac{e^{-t+(2n\pi i + t)s}}{2n\pi i + t}\right]_{s=0}^{s=1} + \frac{2}{1 + t}\left[\frac{(2s-1)e^{2n\pi i s}}{2n\pi i}\right]_{s=0}^{s=1}$$

$$= -\frac{2}{2n\pi i - t} - \frac{2}{2n\pi i + t} - \frac{2i}{n\pi(1 + t)}$$

$$= \frac{8n\pi i}{4n^2\pi^2 + t^2} - \frac{2i}{n\pi(1 + t)},$$

so that

$$\left[\frac{e^{-st} - e^{-(1-s)t}}{1 - e^{-t}} + \frac{2s - 1}{1 + t}\right] \cdot \frac{1}{t} = \sum_{n=1}^{\infty} c_n(t) \frac{\sin 2n\pi s}{n\pi}, \tag{151}$$

$$0 \leqq s \leqq 1, \qquad t \geqq 0$$

where

$$c_n(t) = \frac{1}{t}\left(\frac{8n^2\pi^2}{4n^2\pi^2 + t^2} - \frac{2}{1 + t}\right) = \frac{2}{1 + t} - \frac{2t}{4n^2\pi^2 + t^2} \tag{152}$$

increases with n for any fixed positive value of t. We shall now prove that it is permissible to substitute (151) in (150) and integrate term by term, or in other words, that

$$\int_0^\infty \left(\sum_{\nu=n+1}^{\infty} c_\nu(t) \frac{\sin 2\nu\pi s}{\nu\pi}\right) dt \to 0 \qquad \text{as} \qquad n \to \infty$$

uniformly for $\epsilon \leqq s \leqq 1 - \epsilon$. Writing

$$S_\nu = \sum_{\mu=\nu}^{\infty} \frac{\sin 2\mu\pi s}{\pi\mu},$$

so that

$$\frac{\sin 2\nu\pi s}{\nu\pi} = S_\nu - S_{\nu+1},$$

it follows from (38) that for $\epsilon \leqq s \leqq 1 - \epsilon$

$$|S_\nu| \leqq \frac{1}{\nu\pi \sin \pi\epsilon},$$

and from the identity of summation by parts

$$\sum_{\nu=n+1}^{n+p} c_\nu(S_\nu - S_{\nu+1}) = \sum_{\nu=n+1}^{n+p} (c_\nu - c_{\nu-1})S_\nu + c_n S_{n+1} - c_{n+p+1} S_{n+p+1}$$

we conclude, since $c_\nu(t) - c_{\nu+1}(t) > 0$ for $t > 0$, that

$$(153)\quad \left|\sum_{\nu=n+1}^{n+p} c_\nu(t)\frac{\sin 2\nu\pi s}{\nu\pi}\right| \leqq \frac{1}{\pi\sin\pi\epsilon}\left[\sum_{\nu=n+1}^{n+p}(c_\nu(t)-c_{\nu-1}(t))\cdot\frac{1}{\nu} + \frac{|c_n(t)|}{n+1}+\frac{|c_{n+p+1}(t)|}{n+p+1}\right].$$

From (152) we obtain

$$(154)\quad \int_0^\infty c_n(t)dt = \left[\log\frac{(1+t)^2}{4n^2\pi^2+t^2}\right]_{t=0}^{t=\infty} = 2\log 2n\pi,$$

and since $c_n(t) \gtreqless 0$ as $t \lesseqgtr 4n^2\pi^2$,

$$\int_0^\infty |c_n(t)|\,dt = \int_0^{4n^2\pi^2} c_n(t)dt - \int_{4n^2\pi^2}^\infty c_n(t)dt$$

$$= 2\log\left(1+\frac{1}{4n^2\pi^2}\right) + 2\log 2n\pi < 4\log 2n\pi,$$

so that, by (153)

$$\int_0^\infty\left|\sum_{\nu=n+1}^{n+p} c_\nu(t)\frac{\sin 2\nu\pi s}{\nu\pi}\right|dt < \frac{2}{\pi\sin\pi\epsilon}\left[\sum_{\nu=n+1}^{n+p}\frac{1}{\nu}\log\left(1+\frac{1}{\nu-1}\right) + \frac{2\log 2n\pi}{n+1}+\frac{2\log 2(n+p+1)\pi}{n+p+1}\right].$$

Letting $p\to\infty$, and observing that $\log\left(1+\dfrac{1}{\nu-1}\right) < \dfrac{1}{\nu-1}$ and $\displaystyle\sum_{\nu=n+1}^{\infty}\frac{1}{\nu(\nu-1)} = \frac{1}{n}$, we finally see that

$$\left|\int_0^\infty\left(\sum_{\nu=n+1}^{\infty} c_\nu(t)\frac{\sin 2\nu\pi s}{\nu\pi}\right)dt\right| < \frac{2}{\pi\sin\pi\epsilon}\left(\frac{1}{n}+\frac{2\log 2n\pi}{n+1}\right)\to 0 \text{ as } n\to\infty$$

uniformly for $\epsilon \leqq s \leqq 1-\epsilon$. Substituting (151) in (150) and integrating term by term with the aid of (154), we find for $0 < s < 1$

$$\log\frac{\Gamma(s)}{\Gamma(1-s)} + C(2s-1) = 2\sum_{n=1}^{\infty}\frac{\log n + \log 2\pi}{n\pi}\sin 2n\pi s,$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1-\epsilon$. Using (40) and (121) (the latter being identical with J 6), we now immediately obtain Kummer's series (113).

Additional note. Application of complex integration to the theory of the Gamma function. We observed at the beginning of chapter III that when the Gamma function is defined by means of Euler's integral of the second kind, we are thereby restricted to the region of convergence of this integral, viz., the half plane $\Re(s) > 0$. This difficulty is overcome by the use of complex integration and other more advanced parts of the

theory of functions of a complex variable. The author's original intention to devote a fourth chapter to an exposition of these methods had to be abandoned through lack of space, and it was found feasible only to give the following brief references to the literature.

A paper by Birkhoff[31] contains what is probably the shortest way of arriving at the main properties of the Gamma function from the function theoretic point of view. Starting from the definition

$$\Gamma(s) = \lim_{n\to\infty} \frac{\varphi(s+n+1)}{s(s+1)\cdots(s+n)},$$

$$\varphi(s) = s^{s-1/2}e^{-s}\sqrt{2\pi}, \qquad |\text{arc } s| < \pi,$$

which is essentially equivalent to (J 15), Birkhoff first proves the formula

$$(J\ 14) \qquad \log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} + \omega(s)$$

with the Gudermann expansion of $\omega(s)$ (J 13) and the fundamental property of this function, viz. $|\omega(s)| < K/\rho$, where K is a constant and ρ the distance of s from the nearest point on the negative real axis. Next, the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ (J 6), the infinite products for $\Gamma(s)$ (J 3 and 5), and Euler's integral of the first kind are established, and after a digression on $\psi(s) = \log s - \omega^*(s)$, $|\omega^*(s)| < K/\rho^2$, Birkhoff's paper closes with the expression of Euler's integral of the first kind in terms of the Gamma function (71 in the present paper).

By the use of Cauchy's integral, a systematic study may be made of a whole class of summation formulas, containing that of Euler, and as an application, the formulas (76) and (82), Binet's integral (97) and those of Schaar (143, 147) may be derived from a common point of view, together with the various asymptotic expansions to which they lead. An excellent exposition of this aspect of the theory is given by Lindelöf.[32]

Attention is finally called to an important paper by Mellin,[33] in which is set forth the intimate connection of the Gamma function with a large class of other transcendental functions (for instance, the Riemann Zeta function). The main purpose of Mellin's paper is however to apply the Gamma function to the systematic solution of a class of linear difference equations (the simplest case of which is solved in J § 11) as well as to a corresponding class of linear differential equations, the coefficients of which are polynomials of the first degree.

[31] Birkhoff, G. D., Note on the Gamma function, Bulletin Am. Math. Soc., vol. 20 (1913–14), pp. 1–10.

[32] Lindelöf, E., Le calcul des résidus, Paris, Gauthier-Villars, 1905 (Collection Borel).

[33] Mellin, H., Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen, Math. Annalen, vol. 68 (1910), pp. 305–337.

INVARIANTS WHICH ARE FUNCTIONS OF PARAMETERS OF THE TRANSFORMATION.*

By Oliver E. Glenn.

Among invariant theories of importance in geometry are those leading to concomitants which are functions of one or more of the coefficients of the transformations. An example is the case of invariants of axial rotations in a plane.† A general theory of such concomitants is here considered, for both binary and ternary linear transformations. What is usually true of a general analytical algorithm, ruling various special situations, is exemplified, i. e., the algorithm points to a natural mode of procedure in the investigation of any particular instance under the theory. Application is made in Section II to a theory of the invariants of relativity.

I. Theory of Binariants.

1. Concomitants which are irrational in the parameters. The transformation

$$T: \begin{cases} x_1 = \alpha_1 x_1' + \alpha_2 x_2', \\ x_2 = \beta_0 x_1' + \beta_1 x_2', \end{cases}$$

where the parameters are arbitrary, has two poles which are the roots of the respective linearly independent linear quantics‡

$$(1) \qquad f_{\pm 1} = 2\beta_0 x_1 + (\beta_1 - \alpha_1 \pm \Delta)x_2 \quad (\Delta = \sqrt{(\beta_1 - \alpha_1)^2 + 4\beta_0\alpha_2}).$$

These quantics are covariants of T, and they satisfy the invariant relations

$$(2) \qquad f_1' = \rho_1^{-1} f_1, \qquad f_{-1}' = \rho_{-1}^{-1} f_{-1},$$

in which

$$\rho_{\pm 1} = \tfrac{1}{2}(\beta_1 + \alpha_1 \pm \Delta), \qquad \rho_1 \rho_{-1} = \alpha_1\beta_1 - \alpha_2\beta_0 \equiv D.$$

The binary form of order m

$$f = (a_0, a_1, \cdots, a_m \between x_1, x_2)^m$$

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† Cf. Boole, Cambridge Math. Journal, vol. 3 (1843); Elliott, *Algebra of Quantics* (First ed.), Chap. 15.

‡ Transactions Amer. Math. Society, vol. 18 (1917), p. 450.

has a unique expansion in terms of f_1, f_{-1} as arguments:

$$f = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i} f_1^{m-i} f_{-1}^{i}, \tag{3}$$

concerning which we prove the following

THEOREM. *The linearly independent functions* $\varphi_{m-2i}(i = 0, \cdots, m)$, *linear in* $a_0, \cdots, a_m$, *taken with* f_1, f_{-1}, *compose a complete system of concomitants of* f *under* T *in the domain* $R(1, T, \Delta)$ *to which* f_1, f_{-1} *belong.*

The independence of the expressions φ is evident. For, were they linearly connected, so would be the $m + 1$ linear functions of them, $a_0, \cdots, a_m$, but these are arbitrary.

Let f', the transformed of f by T, be expanded in terms of the arguments

$$f_{\pm 1}' = 2\beta_0 x_1' + (\beta_1 - \alpha_1 \pm \Delta) x_2';$$

$$f' = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i}' f_1'^{m-i} f_{-1}'^{i}.$$

Then the function φ_{m-2i}' is evidently the same function of $a_0', \cdots, a_m'$ that φ_{m-2i} is of $a_0, \cdots, a_m$; moreover, if we apply the inverse of T to f' and use the relations (2) we get

$$f = \sum_{i=0}^{m} \binom{m}{i} \varphi_{m-2i}' \rho_1^{2i-m} D^{-i} f_1^{m-i} f_{-1}^{i},$$

an expansion which must be identical with (3) since (3) is unique. Hence the expressions φ_{m-2i} are invariants of f under T satisfying the invariant relations

$$\varphi_{m-2i}' = \rho_1^{m-2i} D^{i} \varphi_{m-2i} \quad (i = 0, \cdots, m).$$

Any concomitant of f under T can be expressed in terms of $f_1, f_{-1}, \varphi_{m-2i}$ $(i = 0, \cdots, m)$ by means of (1) and the inverse of the following $m + 1$ linear substitutions on $a_0, \cdots, a_m$:

$$\varphi_{m-2i} = \varphi_{m-2i}(a_0, a_1, \cdots, a_m) \quad (i = 0, \cdots, m),$$

hence the theorem is proved.

2. Systems belonging to the domain of rational polynomials. If we seek a fundamental system of concomitants which belong to the domain $R(1, T, 0)$ of rational polynomials in $a_0, \cdots, a_m, x_1, x_2$ and the parameters of the transformation, we will be concerned with linear expressions in terms such as

$$I_r = \varphi_{m-2i_1}^{r_{i_1}} \varphi_{m-2i_2}^{r_{i_2}} \cdots f_1^{s_1} f_{-1}^{s_2}.$$

The invariant relation for I_r is

$$I_r' = \rho_1^{\Sigma r_i(m-2i) - s_1 + s_2} D^{\Sigma r_i i - s_2} I_r, \tag{4}$$

and a linear expression ψ in terms I_r is a concomitant if and only if $\Sigma r_i(m - 2i) - s_1 + s_2$ as well as $\Sigma r_i i - s_2$ are the same for every term. A necessary condition that ψ belong to $R(1, T, 0)$ is that, for each of its terms,

$$
\begin{aligned}
\nu &= \Sigma r_i(m - 2i) - s_1 + s_2 \\
&= r_{i_1}(m - 2i_1) + r_{i_2}(m - 2i_2) + \cdots - s_1 + s_2 = 0.
\end{aligned} \tag{5}
$$

This amounts to a sufficient condition also, for, although not all products I_r are rational, those for which $\nu = 0$ can be arranged in conjugate pairs (I_r, I_{-r}) and the power of D in the invariant relation for I_{-r} will be the same as the power for I_r. In fact,

$$I_{-r} = \varphi_{-(m-2i_1)}{}^{r_{i_1}} \varphi_{-(m-2i_2)}{}^{r_{i_2}} \cdots f_1{}^{s_2} f_{-1}{}^{s_1},$$

for which the power of D is $D^{\Sigma r_i(m-i)-s_1}$. But, if $\nu = 0$,

$$\Sigma r_i(m - i) - s_1 = \Sigma r_i i - s_2.$$

We can now replace the terms of the pair (I_r, I_{-r}) by $I_r + I_{-r}$, $I_r - I_{-r}$, respectively, and the latter binomials belong, essentially, to $R(1, T, 0)$.

Any concomitant multinomial is here reducible in terms of invariant monomials. The question of the finiteness of a complete system, as well as the problem of determining it explicitly are, therefore, solved by a known lemma due to Hilbert, viz.: If an infinite system of monomials in n letters be formed according to any law sufficiently definite to locate an arbitrarily chosen monomial within or without the system, then there will always exist, within the system, a finite set of monomials such that every monomial of the system is divisible by at least one of the set.

In the present case the letters involved are φ_{m-2i} $(i = 0, \cdots, m)$, f_1, f_{-1}, and $n = m + 3$, while the law by which the system is formed is embodied in the linear diophantine equation

$$
\begin{aligned}
&s_2 + r_0 m + r_1(m - 2) + r_2(m - 4) + \cdots \\
&\qquad = \cdots + r_{m-2}(m - 4) + r_{m-1}(m - 2) + r_m m + s_1,
\end{aligned} \tag{6}
$$

which is to be satisfied in positive integers (including zero) r_i, s_j. The terms on the two sides of this equation, adjacent to the equality sign, are $2r_{\frac{1}{2}m-1}$, $2r_{\frac{1}{2}m+1}$ if m is even, and $r_{\frac{1}{2}(m-1)}$, $r_{\frac{1}{2}(m+1)}$ if m is odd, and in the former case φ_0 exists and is included in the irreducible system.

We have now proved the following

THEOREM. *A fundamental system of concomitants of f under T in $R(1, T, 0)$ is given by the irreducible solutions of the linear diophantine equation (6). The number of concomitants in the system is equal to the finite number of these irreducible solutions, increased, if m is even, by unity.*

The enumeration of particular complete systems for special values of m is the same for the situation treated in section II as in the present general theory, and these details are omitted to be taken up in that connection. We add, also, that this theory holds if the quantity under the radical Δ is a square, so that Δ is only apparently an irrationality, but does not hold if $\Delta = 0$.

II. The Instance of Einstein's Relativity Transformations.

1. The transformations. Two moving systems of reference S and S' are conceived, which, for the sake of concreteness, may be taken to be two platforms on each of which are installed instruments for taking measurements, such as clocks for measuring time, rules for measuring lengths, and so on. Let these two systems have the relative velocity v in the line l. Suppose that systems of rectangular coördinates are attached to S and S' in such a way that the x-axis of each system is in the line l, and let the y-axis and the z-axis on S be parallel, respectively, to the y'-axis and the z'-axis on S'. Supposing the origins to coincide at the time $t = 0$, let the coördinates on S be denoted by x, y, z, t, and those referring to S' by x', y', z', t'. Then, as was shown first by Einstein,

$$(7) \qquad t = \mu(c^2t' + vx')/c, \quad x = \mu(vt' + x')c, \quad y = y', \quad z = z',$$

where

$$\mu = 1/\sqrt{c^2 - v^2},$$

and c is the velocity of light.*

Let the first two equations of (7) be denoted by τ, then τ is unitary, and when we treat its invariant theory as a special situation under section I we have $D = 1$. Instead of distinguishing the two types of concomitants with respect to domains of rationality we now refer to them merely as non-absolute (relative) systems and absolute systems.

When (1) and (2) are particularized to correspond to τ we find that f_1, f_{-1} become, essentially,

$$(8) \qquad \xi = ct + x, \qquad \eta = ct - x.$$

These are universal covariants of τ for all values of the relative velocity v, for which the invariant relations are

$$\xi' = \rho^{-1}\xi, \qquad \eta' = \rho\eta,$$

where

$$\rho = \sqrt{\frac{c + v}{c - v}}.$$

* Einstein, Annalen der Physik, vol. 17 (1905). Lorentz, Einstein and Minkowski, *Das Relativitätsprinzip* (1913), p. 27. R. D. Carmichael, *The Theory of Relativity* (1913), p. 44.

Let

$$f = a_0 t^m + m a_1 t^{m-1} x + \cdots + a_m x^m,$$

where $a_0, \cdots, a_m$ are constants, or arbitrary functions of c, y, z, then the expansion (3) is given by substituting, in f, from the inverse of (8), i. e.,

$$(9) \qquad t = (\xi + \eta)/2c, \qquad x = c(\xi - \eta)/2c.$$

As it is evident that φ_{m-2i} contains the constant $(2c)^m$ in the denominator we write

$$(2c)^m \varphi_{m-2i} = \psi_{m-2i},$$

whence follows

$$\psi_{m-2i}' = \rho^{m-2i} \psi_{m-2i} \ (i = 0, \cdots, m).$$

The concomitants are here functions of the parameters in τ but they are, in fact, free from the relative velocity of the systems of reference.

We note that these systems become orthogonal invariant systems when the velocity of light c is replaced by $\sqrt{-1}$, the covariant $-\xi\eta$ then becoming the *absolute*, $x^2 + t^2$.

2. Calculation of the non-absolute system of f. The invariants ψ_{m-2i} can be derived in explicit form when m is general. Noting that

$$(\xi + \eta)^h (\xi - \eta)^k = \sum_{i=0}^{h+k} \left[\sum_{t=0}^{i} (-1)^t \binom{h}{i-t} \binom{k}{t} \right] \xi^{h+k-i} \eta^i,$$

it is evident that the transformed of

$$f = \sum_{j=0}^{m} \binom{m}{j} a_j t^{m-j} x^j,$$

by (9), is

$$(10) \qquad f = \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{t=0}^{i} (-1)^t \binom{m}{j} \binom{m-j}{i-t} \binom{j}{t} a_j c^j \xi^{m-i} \eta^i / (2c)^m.$$

Hence

$$(11) \qquad \binom{m}{i} \psi_{m-2i} = \sum_{j=0}^{m} \sum_{t=0}^{i} (-1)^t \binom{m}{j} \binom{m-j}{i-t} \binom{j}{t} a_j c^j \quad (i = 0, \cdots, m).$$

A few special cases of this formula are added to facilitate writing down particular systems. The invariant $\psi_{-(m-2i)}$ may be obtained from ψ_{m-2i} by changing the signs of all odd powers of c in the latter, i. e., the ψ's at equal distances from the ends of expansion (10) are conjugates. Hence the four ψ's written below suffice to give at once the explicit complete systems for all orders up to $m = 7$ inclusive.

$$\psi_m = a_0 + m a_1 c + \binom{m}{2} a_2 c^2 + \cdots + \binom{m}{j} a_j c^j + \cdots + a_m c^m,$$

$$1! \binom{m}{1} \psi_{m-2} = m a_0 + \binom{m}{1} (m-2) a_1 c + \binom{m}{2} (m-4) a_2 c^2 + \cdots$$
$$+ \binom{m}{j} (m - 2j) a_j c^j + \cdots - m a_m c^m,$$

$$2!\binom{m}{2}\psi_{m-4} = m(m-1)a_0 + \binom{m}{1}(m-1)(m-4)a_1c$$
$$+ \binom{m}{2}(m^2 - 9m + 16)a_2c^2 + \cdots$$
$$+ \binom{m}{j}(m^2 - \overline{4j+1}\,m + 4j^2)a_jc^j + \cdots$$
$$+ m(m-1)a_mc^m,$$

$$3!\binom{m}{3}\psi_{m-6} = m(m-1)(m-2)a_0 + \binom{m}{1}(m-1)(m-2)(m-6)a_1c + \cdots$$
$$+ \binom{m}{j}[m^3 - 3(2j+1)m^2 + 2(6j^2 + 3j + 1)m$$
$$- 4(2j^3 + j)]a_jc^j + \cdots - m(m-1)(m-2)a_mc^m,$$

et cetera.

3. Absolute invariants of relativity. A complete system of absolute concomitants of f under τ is constructed from the irreducible solutions of (6). Thus, if $m = 2$, the sets of values of s_2, r_0, r_2, s_1 respectively, in these solutions, are 1, 0, 0, 1; 0, 1, 1, 0; 0, 1, 0, 2; 2, 0, 1, 0. Hence the system for the quadratic is

$$(12) \qquad \gamma : \psi_0, \quad J = \xi\eta, \quad \delta : \psi_2\psi_{-2}, \quad \epsilon_{\pm 1} : \psi_2\xi^2 \pm \psi_{-2}\eta^2,$$

where a colon is used instead of the equality sign to indicate that irrelevant constant factors are to be deleted.

We arrange these solutions in tables as below (cf. table for $m = 2$), juxtaposing under a double notation, like $\epsilon_{\pm 1}$, the solutions representing conjugate products. If a solution is symmetrically placed with reference to the median line of the table the corresponding product is its own conjugate. We omit writing the systems in the form (12) as all types can easily be read off from the tables.

The number of concomitants in the absolute system of a quartic is 12.

$m = 1.$

	s_2	r_0	r_1	s_1
J	1	0	0	1
α	0	1	1	0
$\beta_{\pm 1}$	0	1	0	1
	1	0	1	0

$m = 2.$

	s_2	r_0	r_2	s_1
J	1	0	0	1
δ	0	1	1	0
$\epsilon_{\pm 1}$	0	1	0	2
	2	0	1	0

$m = 3.$

	s_2	r_0	r_1	r_2	r_3	s_1
J	1	0	0	0	0	1
ζ	0	1	0	0	1	0
η	0	0	1	1	0	0
$\theta_{\pm 1}$	0	1	0	3	0	0
	0	0	3	0	1	0
$\iota_{\pm 1}$	0	0	1	0	0	1
	1	0	0	1	0	0
$\kappa_{\pm 1}$	0	1	0	0	0	3
	3	0	0	0	1	0
$\lambda_{\pm 1}$	0	1	0	1	0	2
	2	0	1	0	1	0
$\mu_{\pm 1}$	0	1	0	2	0	1
	1	0	2	0	1	0

The actual invariants and covariants represented in these tables are now given (cf. 11 and 12).

$m = 1.$

$$J = c^2t^2 - x^2, \quad \alpha = a_0^2 - c^2a_1^2, \quad \beta_{+1} = f, \quad \beta_{-1} = c^2a_1t + a_0x.$$

$m = 2.$

$$J, \quad \gamma = a_0 - c^2a_2, \quad \delta = a_0^2 - 4c^2a_1^2 + c^4a_2^2 + 2c^2a_0a_2,$$

$$\epsilon_{+1} = (a_0 + c^2a_2)(c^2t^2 + x^2) + 4c^2a_1tx,$$

$$\epsilon_{-1} = a_1(c^2t^2 + x^2) + (a_0 + c^2a_2)tx.$$

$$m = 3.$$

$$J,\ \zeta = a_0^2 - 9c^2a_1^2 + 9c^4a_2^2 - c^6a_3^2 + 6c^2a_0a_2 - 6c^4a_1a_3,$$

$$\eta = a_0^2 - c^2a_1^2 + c^4a_2^2 - c^6a_3^2 - 2c^2a_0a_2 + 2c^4a_1a_3,$$

$$\theta_{\pm 1} = (a_0 + 3ca_1 + 3c^2a_2 + c^3a_3)(a_0 - ca_1 - c^2a_2 + c^3a_3)^3$$
$$\pm (a_0 - 3ca_1 + 3c^2a_2 - c^3a_3)(a_0 + ca_1 - c^2a_2 - c^3a_3)^3,$$

$$\iota_{+1} = (a_0 - c^2a_2)t + (a_1 - c^2a_3)x,$$

$$\iota_{-1} = (a_1 - c^2a_3)c^2t + (a_0 - c^2a_2)x,$$

$$\kappa_{+1} = (a_0 + 3c^2a_2)(c^2t^3 + 3tx^2) + (3a_1 + c^2a_3)(3c^2t^2x + x^3),$$

$$\kappa_{-1} = (3a_1 + c^2a_3)(c^4t^3 + 3c^2tx^2) + (a_0 + 3c^2a_2)(3c^2t^2x + x^3),$$

$$\lambda_{+1} = A(c^2t^2 + x^2) + 4Bc^2tx, \quad \lambda_{-1} = B(c^2t^2 + x^2) + Atx,$$

$$\mu_{+1} = Ct + Dx, \quad \mu_{-1} = Dc^2t + Cx,$$

where

$$A = a_0^2 - 3c^2a_1^2 - 3c^4a_2^2 + c^6a_3^2 + 2c^2a_0a_2 + 2c^4a_1a_3,$$

$$B = a_0a_1 + c^2a_0a_3 - 3c^2a_1a_2 + c^4a_2a_3,$$

$$C = a_0^3 + 3c^6a_2^3 - 5c^2a_0a_1^2 - 5c^4a_0a_2^2 + 3c^6a_0a_3^2 + c^2a_0^2a_2 + 9c^4a_1^2a_2$$
$$+ c^8a_2a_3^2 + 2c^4a_0a_1a_3 - 10c^6a_1a_2a_3,$$

$$D = a_0^2a_1 + 3c^2a_0^2a_3 + 9c^4a_1a_2^2 - 5c^4a_1^2a_3 + c^6a_1a_3^2 - 5c^6a_2^2a_3$$
$$+ 2c^4a_0a_2a_3 - 10c^2a_0a_1a_2 + 3c^2a_1^3 + c^8a_3^3.$$

A single syzygy connects the quantics of each of the first two of these systems, as follows:

$$\Sigma_1 = \alpha J - c^2f^2 + \beta_{-1}^2 = 0,$$

$$\Sigma_2 = \delta J^2 - \epsilon_{+1}^2 + 4c^2\epsilon_{-1}^2 = 0.$$

III. Ternariants.

The general ternary transformations

$$S: \begin{cases} x = l_1x' + l_2y' + l_3z', \\ y = m_1x' + m_2y' + m_3z', \\ z = n_1x' + n_2y' + n_3z', \end{cases}$$

have three poles in a plane, and the linear ternary quantics representing the lines joining these poles in pairs, are covariants of S. In fact, assuming

that S transforms

$$f = a_1x + a_2y + a_3z \tag{13}$$

into kf' we readily find that k is a root of the characteristic equation

$$k^3 - \Sigma_1 k^2 + \Sigma_2 k - D = 0, \tag{14}$$

where Σ_i $(i = 1, 2)$ is the sum of all of the principal minors of order i, of the determinant D of S.

1. Invariants under rotations of three-dimensional axes. If S is the transformation which rotates a set of rectangular axes in the three-space into another rectangular system with the same origin, then, l_j, m_j, n_j are direction cosines connected by a variety of well-known relations. Then $D = \pm 1$, and the equation (14) can be reduced to

$$k^3 - ak^2 + \sigma ak - \sigma = 0, \tag{15}$$

where $\sigma = \operatorname{sgn} D$; i. e., is $+1$ or -1 according as D is $+1$ or -1; and a is the sum of the direction cosines occurring in the principal diagonal of D. Without loss of generality we can now substitute

$$a = \sigma + 2\sigma \cos \theta,$$

θ being an auxiliary angle, whence the three roots of (15) become σ, $\sigma e^{i\theta}$, $\sigma e^{-i\theta}$ $(i = \sqrt{-1})$. Replacing k, in $kf' = f$, by these three values, in succession, and solving for the ratios $a_1 : a_2 : a_3$, we find the three linear covariants of S to be

$$\begin{aligned} f_{\pm 1}{}^{(\sigma)} &= (l_3 + \sigma n_1 e^{\pm i\theta})x + (m_3 + \sigma n_2 e^{\pm i\theta})y + (n_3 - \overline{l_1 + m_2}\sigma e^{\pm i\theta} + e^{\pm 2i\theta})z, \\ f_0{}^{(\sigma)} &= (l_3 + \sigma n_1)x + (m_3 + \sigma n_2)y + (n_3 - \overline{l_1 + m_2}\sigma + 1)z. \end{aligned} \tag{16}$$

Three linear contravariants, representing the poles, are the eliminants of these three covariants taken, in pairs, with $ux + vy + wz$, but contragrediency is equivalent to cogrediency under S.

The general ternary quantic of order m,

$$f(x, y, z) = \Sigma \frac{m!}{r!\,s!\,t!} a_{rst} x^r y^s z^t \quad (r + s + t = m),$$

has a unique expansion* in terms of $f_{+1}{}^{(\sigma)}$, $f_{-1}{}^{(\sigma)}$, $f_0{}^{(\sigma)}$ as argument forms:

$$f = \sum_{t=0}^{m} \binom{m}{t} \sum_{i=0}^{m-t} \binom{m-t}{i} \varphi^{(t,\sigma)}_{m-t-2i} f_{+1}{}^{(\sigma)m-t-i} f_{-1}{}^{(\sigma)i} f_0{}^{(\sigma)t}.$$

The $\frac{1}{2}(m + 1)(m + 2)$ coefficient forms φ are invariants of f under S, linear in the coefficients a_{rst}, and belonging to the domain of complex

* Transactions Amer. Math. Society, vol. 15 (1914), p. 82.

numbers. They satisfy the invariant relations

$$\varphi_{m-t-2i}^{(t,\sigma)\prime} = \rho^{m-t-2i}\varphi_{m-t-2i}^{(t,\sigma)} \quad \begin{pmatrix} t = 0, \cdots, & m \\ i = 0, \cdots, & m - t \end{pmatrix}, \tag{17}$$

in which $\rho = \sigma e^{i\theta}$. The theorem below follows:

THEOREM. *A complete system of relative concomitants of f, under the transformations S of determinant σ, is composed of $f_{\pm 1}^{(\sigma)}$, $f_0^{(\sigma)}$, and the $\frac{1}{2}(m+1)(m+2)$ invariants $\varphi_{m-t-2i}^{(t,\sigma)}$.*

In order to construct explicitly the invariants (17), it is convenient to solve for the inverse of the system of equations (16) and substitute the resulting linear expressions in $f_{\pm 1}^{(\sigma)}$, $f_0^{(\sigma)}$, which are the values of x, y, z, in f.

The determinant of $f_{+1}^{(\sigma)}$, $f_{-1}^{(\sigma)}$, $f_0^{(\sigma)}$ can be put in the form

$$\Delta = \sigma(l_3 n_2 - m_3 n_1)(1 - e^{i\theta})(1 - e^{-i\theta})(e^{-i\theta} - e^{i\theta}),$$

and the solution of (16), and use of relations among direction cosines, gives

$$\Delta x = A_{-1} f_{+1}^{(\sigma)} - A_{+1} f_{-1}^{(\sigma)} + A_0 f_0^{(\sigma)},$$

$$\Delta y = B_{-1} f_{+1}^{(\sigma)} - B_{+1} f_{-1}^{(\sigma)} + B_0 f_0^{(\sigma)},$$

$$\Delta z = C_{-1} f_{+1}^{(\sigma)} - C_{+1} f_{-1}^{(\sigma)} + C_0 f_0^{(\sigma)},$$

in which

$$A_{\pm 1} = (e^{\pm i\theta} - 1)[\sigma(l_1 m_3 - l_2 l_3) - \sigma n_2 e^{\pm i\theta} - m_3(e^{\pm i\theta} + 1)],$$

$$B_{\pm 1} = -(e^{\pm i\theta} - 1)[\sigma(m_2 l_3 - m_1 m_3) - \sigma n_1 e^{\pm i\theta} - l_3(e^{\pm i\theta} + 1)],$$

$$C_{\pm 1} = (e^{\pm i\theta} - 1)\sigma(m_3 n_1 - l_3 n_2),$$

$$A_0 = (e^{i\theta} - e^{-i\theta})[\sigma(l_1 m_3 - l_2 l_3) - \sigma n_2 - m_3(e^{i\theta} + e^{-i\theta})],$$

$$B_0 = -(e^{i\theta} - e^{-i\theta})[\sigma(m_2 l_3 - m_1 m_3) - \sigma n_1 - l_3(e^{i\theta} + e^{-i\theta})],$$

$$C_0 = (e^{i\theta} - e^{-i\theta})\sigma(m_3 n_1 - l_3 n_2).$$

Through substitution in f, we obtain, by a well-known principle,

$$\varphi_m^{(0,\sigma)} = f(A_{-1}, B_{-1}, C_{-1})/\Delta^m,$$

and hence

$$\varphi_{m-t-2i}^{(t,\sigma)} = \Delta_1^i \Delta_0^t \varphi_m^{(0,\sigma)} \Big/ \frac{|m}{|m-i-t} \quad \begin{pmatrix} t = 0, \cdots, m \\ i = 0, \cdots, m - t \end{pmatrix},$$

where

$$\Delta_1 = -A_{+1}\frac{\partial}{\partial A_{-1}} - B_{+1}\frac{\partial}{\partial B_{-1}} - C_{+1}\frac{\partial}{\partial C_{-1}},$$

$$\Delta_0 = A_0\frac{\partial}{\partial A_{-1}} + B_0\frac{\partial}{\partial B_{-1}} + C_0\frac{\partial}{\partial C_{-1}}.$$

2. **Absolute concomitants.** For a fixed value of t $\varphi^{(t,\sigma)}_{-(m-t-2i)}$ is conjugate to $\varphi^{(t,\sigma)}_{m-t-2i}$ $(i = 0, \cdots, m - t)$; hence if

$$I_r = \prod_{t=0}^{m} \varphi^{(t,\sigma)^{r_{i_1}^{(t)}}}_{m-t-2i_1} \varphi^{(t,\sigma)^{r_{i_2}^{(t)}}}_{m-t-2i_2} \cdots f_{+1}{}^{(\sigma)^{s_1}} f_{-1}{}^{(\sigma)^{s_2}} f_0{}^{(\sigma)^{s_3}}$$

is any product from the relative system, the conjugate product is

$$I_{-r} = \prod_{t=0}^{m} \varphi^{(t,\sigma)^{r_{i_1}^{(t)}}}_{-(m-t-2i_1)} \varphi^{(t,\sigma)^{r_{i_2}^{(t)}}}_{-(m-t-2i_2)} \cdots f_{+1}{}^{(\sigma)^{s_2}} f_{-1}{}^{(\sigma)^{s_1}} f_0{}^{(\sigma)^{s_3}},$$

and the exponent of ρ in the invariant relation for I_r is

$$E = \sum_{t=0}^{m} \sum_{i=0}^{m-t} r_i^{(t)}(m - t - 2i) - s_1 + s_2.$$

The concomitants $I_r \pm I_{-r}$, deprived of constant factors, are both real. Application of Hilbert's lemma to the theory of absolute concomitants, as in I, 2, gives, therefore:

Theorem. *A fundamental system of absolute concomitants of f, under the rotational transformations of determinant σ, is given by the finite set of irreducible solutions in positive integers $r_i^{(t)}$, s_j (zero values being included) of the linear diophantine equation $E = 0$. The number of concomitants in the system is equal to the number of irreducible solutions increased by $\frac{1}{2}(m + 4)$ if m is even and by $\frac{1}{2}(m + 3)$ if m is odd.*

The numbers $\frac{1}{2}(m + 4)$, $\frac{1}{2}(m + 3)$, added to the number of irreducible solutions, correspond to the existent absolute invariants $f_0{}^{(\sigma)}$, $\varphi_0{}^{(t,\sigma)}$, where t is such that $m - t$ is even. These are not furnished by the solution of $E = 0$.

The systems considered in the present section III may be interpreted as invariant systems of the curve $f = 0$ where f is transformed by the ternary substitutions furnished by the formulas for axial rotations in three-space. They may also be interpreted, in the sense of Boole, as systems of invariants of the three-dimensional surface

$$f(x, y, z) = 1,$$

under the operations of rotation of rectangular coördinate axes.

University of Pennsylvania,
September, 1917.

A THEOREM ON EXHAUSTIBLE SETS CONNECTED WITH DEVELOPMENTS OF POSITIVE REAL NUMBERS.*

By Henry Blumberg.

1. We start with any *aggregate* A whatsoever, and deal with "*developments*" of positive real numbers ξ in the form of an infinite sequence

$$(D) \qquad \xi = \{x_1, x_2, \cdots x_n, \cdots\},$$

where x_n belongs to A. By a "*development of* ξ," we here understand simply an infinite sequence of elements of A, *associated* with ξ according to any given law, as yet—but not eventually—unrestricted. The equality sign in (D) signifies nothing more than that the infinite sequence is *associated* with ξ. We assume at once, as the term "development" might indicate, that *two different* ξ's *cannot have identical developments* (D). We do *not* demand that every ξ shall possess at least one development, nor that the development of ξ shall be unique. However, we do suppose that *no* ξ *has more than a finite number of different developments* (D).

If ξ possesses at least one development (D), we call it "*developable*"; otherwise, "*non-developable*."

If the infinite sequence $\{x_1, x_2, \cdots x_n, \cdots\}$ is a development of a real number ξ, we say that it is a "*proper development*"; if no ξ exists that has the sequence as a development, we call it an "*improper development*." In the former case, ξ is called the "*prototype*" of $\{x_1, x_2, \cdots x_n, \cdots\}$.

According to the above, the prototype of a proper development is unique; and the development of a developable number may be multiply but not infinitely "valued."

In the case of the ordinary infinite decimal development, A consists of the set $(0, 1, \cdots 9)$. Here every ξ is developable, but rational numbers may have two developments. A will consist of the set of positive integers, if we represent $\xi (< 1)$ as an infinite continued fraction in the form

$$\xi = \frac{1}{x_1 + \frac{1}{x_2 +}}$$

* Read before the American Mathematical Society, December, 1913. We use (after Denjoy) the term "*exhaustible*"—instead of the phrase, "of first category" (Baire)—to denote a set that is the sum of a countable set of non-dense sets; and later,—see Theorem below—the term "*residual*," to denote the complementary set of an exhaustible set. Cf. Denjoy, Journal de Mathématiques, ser. 7, vol. 1 (1915), pp. 122–125. Instead of restricting ourselves to positive real numbers, we might discuss all real numbers; but the gain would be negligible.

requiring x_n to be a positive integer. In this case, rational numbers are non-developable.

For our purpose, we demand that the development (D) shall possess the

Property P. If in (D), the elements $x_1, x_2, \cdots x_n$ are fixed, and the remaining elements x_ν, $\nu > n$ range in every possible manner over the aggregate A, then the set of prototypes of the proper developments thus obtained constitute the totality of developable numbers in one and the same interval—closed or open on one or both sides—of the linear continuum.*

It is to be understood that such an interval is allowed to extend to ∞, if necessary. We distinctly do not demand the unique existence of such an interval; but there is always a definite "largest,"† which may be obtained from any interval of the described character, by extending the latter as far as possible both to the left and the right, with the restriction that no interior developable numbers shall be allowed that do not belong to the set of prototypes obtained in the definition of P‡. We shall speak of this largest interval as "the interval associated with the sequence $(x_1, x_2, \cdots x_n)$."

Property P is evidently possessed by the ordinary decimal development and by the continued fraction development described above.

The object of the present note is to communicate the following:

THEOREM. *Let (C) be any condition whatsoever that mates—uniquely or not—with every finite sequence $(x_1, x_2, \cdots x_n)$ of elements of A a finite sequence $(x_1', x_2', \cdots x_p')$ of elements of A. Let S be the totality of developable real numbers ξ possessing no development (D) in which it happens infinitely often that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates, $(x_1', x_2', \cdots x_p')$, so that*

$$x_{n+1} = x_1', \qquad x_{n+2} = x_2', \qquad \cdots x_{n+p} = x_p'.$$

Then S is exhaustible, and the complementary set, consisting of non-developable numbers and of those developable numbers that have at least one development in which it happens for an infinite number of values of n that the

* Our desire here is not that of attaining, as can be done without great difficulty, postulates of extreme simplicity. It is rather that of at once formulating the essential property of the developments (D), that is possessed by the well-known developments and is sufficient for the proof of the Theorem of this note.

† Except possibly when the set of prototypes defined in Property P is the null-set; but this case is trivial for the applications of P.

‡ The largest interval may be defined also as the "sum"—in the sense of the Theory of Aggregates—of all possible intervals of the described character.

*sequence of the first n elements is immediately succeeded by one of its mates, is a residual set.**

Proof. Let S_k represent the set of developable numbers that possess no development (D) in which it happens after $n = k$ that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates according to (C). It follows, by the use of the fact that no number has an infinity of different developments, that every element of S belongs to at least one S_k. To prove that S is exhaustible, we shall prove that S_k is a non-dense set. This will be proved, if we show that in every neighborhood of a given developable number ξ, with the development (D), there is an interval free from points of S_k. If ξ is not a limit from both sides of developable numbers, the existence of such an interval is obvious. We may and do assume, therefore, without loss of generality, that ξ is a both-sided limit of developable numbers. With every sequence $(x_1, x_2, \cdots x_n)$, there is associated, according to Property P, a definite interval I_n, containing ξ. As n approaches ∞, the length m_n of I_n must approach zero. For

$$I_1 \supseteqq I_2 \supseteqq I_3 \supseteqq \cdots;$$

and if m_n does not approach zero, I_n approaches a definite interval I—closed or open on one or both sides—containing ξ. As ξ is a both-sided limit of developable numbers, I must contain a developable number $\xi' \neq \xi$. This number, lying in I_n, has as the first n elements of one of its developments the sequence $(x_1, x_2, \cdots x_n)$. Since this holds for every n, and ξ' has no more than a finite number of developments, it follows that at least one development of ξ' is identical with the development (D) of ξ, contrary to the uniqueness of the prototype of a proper development. Consequently,

$$\lim_{n \to \infty} m_n = 0.$$

Let now $n > k$, and let the sequence $(x_1', x_2', \cdots x_p')$ be a mate of the sequence $(x_1, x_2, \cdots x_n)$ according to (C). The interval I_n' associated with the sequence $(x_1, x_2, \cdots x_n, x_1', x_2', \cdots x_p')$ contains no element of S_k, since I_n' contains only non-developable numbers or developable numbers having at least one development beginning with the sequence $(x_1, x_2, \cdots x_n, x_1', x_2', \cdots x_p')$. Moreover, I_n' is contained in the interval I_n associated with the sequence $(x_1, x_2, \cdots x_n)$. Since, by taking n sufficiently large, the interval I_n may, according to $\lim m_n = 0$, be

* Condition (C) may, in the terminology of E. H. Moore—see his *Introduction to a Form of General Analysis* (1910), preface—be described as a function on M to M, where M stands for the set of finite sequences of elements of A. These finite sequences enter fundamentally in the writer's paper, On the Factorization of Expressions of Various Types, Trans. of the Am. Math. Soc., vol. 17 (1916), pp. 517–544, where they are called "parentheses."

brought within any given neighborhood, however small, of ξ, it follows that the interval I_n', which is free from elements of S_k, may be likewise brought within such a neighborhood. The non-density of S_k is thus established.

Since the decimal development and the continued fraction development described above are special cases of the general development (D), we have the following consequences, formulated, for the sake of simplicity, for the set of numbers between 0 and 1:

COROLLARY 1. *Let* (C) *be any condition that mates with every finite sequence* $(x_1, x_2, \cdots x_n)$, *where* x_ν *is one of the integers* $0, 1, 2, \cdots 9$, *at least one sequence of the same character. Then the set of positive real numbers* < 1, *in whose decimal development the sequence* $(x_1, x_2, \cdots x_n)$ *is immediately followed by one of its mates for only a finite number of values of* n, *is exhaustible.*

COROLLARY 2. *Let the positive real number* ξ (< 1) *be represented as an infinite continued fraction in the form*

$$\cfrac{1}{x_1 + \cfrac{1}{x_2 + \cdots}},$$

the x's being positive integers. Let (C) *be any condition that mates with every finite sequence of positive integers at least one sequence of the same character. Then the set of positive real numbers* < 1, *in whose continued fraction development defined above, the sequence* $(x_1, x_2, \cdots x_n)$ *is immediately followed by one of its mates for only a finite number of values of* n, *is exhaustible.*

There is no difficulty in seeing that *the Theorem applies to all the well-known developments of real numbers.* In addition to those already referred to, we mention the decimal development in the scale of k, the generalized decimal development of Strauss,* and the development as an infinite product, due to Cantor.†

For purposes of illustration, we mention several particular cases of Corollary 1.

(a) Let (C) mate with every finite sequence the integer 1. Then, according to Corollary 1, *the set of decimal fractions in which* 1 *occurs only finitely often is exhaustible.*

(b) Let (C) mate with the sequence $(x_1, x_2, \cdots x_{n-1})$ a sequence of $f_n + 2$ terms, the first and the last being $\neq 1$, and the f_n others $= 1$; here f_n stands for a given function of n taking only positive integral values.

* Acta Math., vol. 11 (1887–88), pp. 13–18.

† See Hobson, "The Theory of Functions of a Real Variable" (1907), p. 48.

Then according to Corollary 1, *the totality of decimal fractions in which it happens infinitely often that an unbroken sequence of* 1*'s commences exactly at the* $(n + 1)$*th place and ends exactly at the* $(n + f_n)$*th place, is a residual set.*

(c) Let n_1 be the number of 1's in the first n figures of the decimal development of ξ. Then as follows readily from (b) or Corollary 1, *the totality of the numbers* ξ *for which* n_1/n *has, as* n *approaches* ∞, *every number between* 0 *and* 1 *as a limit, is a residual set.*

2. Remarks on generalizations and connection with certain results of Hardy and Littlewood. (a) The use of the term "immediately" in the clause, "that the sequence $(x_1, x_2, \cdots x_n)$ is immediately succeeded by one of its mates," involves less of a restriction than may appear at first sight. For example, *there is no gain whatsoever in generality in replacing "immediately succeeded by one of its mates . . ." by "succeeded by a mate* $(x_1', x_2', \cdots x_p')$ *that begins exactly at the* $(n + f_n + 1)$*th place of the development, so that*

$$x_{n+f_n+1} = x_1', \qquad x_{n+f_n+2} = x_2', \qquad \cdots,"$$

where f_n stands for a given function of n independent of condition (C) and taking only positive integral values. For, by using instead of the given condition (C), one that mates with $(x_1, x_2, \cdots x_n)$ every sequence $(x_{n+1}, x_{n+2}, \cdots x_{n+f_n}, x_1', x_2', \cdots x_p')$, where $x_{n+1}, x_{n+2}, \cdots x_{n+f_n}$ take independently all possible values, we are led back precisely to the situation of the Theorem. Again, it may be shown in like manner, that there would be no gain in substituting for the sequence $(x_1, x_2, \cdots x_n)$ the sequence $(x_{\lambda_n}, x_{\lambda_n+1}, \cdots x_{\mu_n})$, where λ_n and μ_n are given functions of the character of f_n described above; nor is anything gained in employing simultaneously both apparent generalizations just described.

(b) The supposition that no ξ possesses more than a finite number of developments may be replaced by the following weaker condition: If $\{x_1, x_2, \cdots x_n, \cdots\}$ is a given proper development, and if ξ has, for every value of n, at least one development beginning with the sequence $(x_1, x_2, \cdots x_n)$ then ξ has the given development as one of its developments. However, if this modification is introduced, the wording of the Theorem should be changed to read "possessing no development (D) in which it happens more than k_ξ times . . ." instead of "possessing . . . infinitely often," where k_ξ is an integer depending on ξ, but not on its various developments; and the latter part of the Theorem should, of course, be correspondingly modified.

(c) The generality of condition (C)—see also the first remark—is such that one is tempted to say, in a loose description of the Theorem, denoting the sequence $(x_1, x_2, \cdots x_n)$ as an "approximate development of

ξ," that "the set of those real numbers that have only a finite number of approximate developments in which a given thing happens, is almost always exhaustible." If we conceive, as has been suggested, of an exhaustible set as one "qualitatively poor" in elements, and a set of measure zero as one "quantitatively poor," the Theorem offers an unexpected contrast with results of Hardy and Littlewood on the measure of certain point sets.* For example, in contrast even to the particular case (c) of Corollary 1, one of the theorems of Hardy and Littlewood states† that *the totality of real numbers between* 0 *and* 1 *for which* n_1/n, *as* n *approaches* ∞, *has the limit* 1/10 *is of measure* 1.

(d) The extension of our Theorem to more general sets than that of the real numbers—in particular, to the set of points in n-space—may be made with only slight modification of our assumptions; but we shall not enter here into the discussion of such extensions.

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* Acta Mathematica, vol. 37 (1914), pp. 155–190, especially p. 183 et seq.

† Loc. cit., p. 186.

SOLUTION OF THE DIFFERENTIAL EQUATION $dx^2 + dy^2 + dz^2 = ds^2$ AND ITS APPLICATION TO SOME GEOMETRICAL PROBLEMS.

By Alexander Pell.

In this paper we give the solution of the equation

$$dx^2 + dy^2 + dz^2 = ds^2, \tag{1}$$

where x, y, and z are functions of a single parameter, in a different form from those given by E. Salkowski* and L. P. Eisenhart.† The new form shows that the coördinates of curves having the same differential element of arc differ from one another by the values of the coördinates of certain minimal curves. The formulæ for the direction cosines of the tangents, principal normals and binormals enable us to give simple solutions of some geometrical problems.

The equations of spherical curves are deduced, in which the differential of arc is expressed rationally.

1. **Parametric Representation of Curves.** If ψ and F are arbitrary analytic functions of a parameter u for a certain domain, an easy computation shows the truth of the following identity

$$\left(\frac{1-u^2}{2}\psi''' - uF''\right)^2 + \left(i\frac{1+u^2}{2}\psi''' + iuF''\right)^2 + (u\psi''' + F'')^2 = F''^2, \tag{2}$$

where the primes indicate the derivatives of the functions ψ and F with respect to u.

If we put

$$\frac{dx}{du} = \frac{1-u^2}{2}\psi''' - uF'', \qquad \frac{dy}{du} = i\frac{1+u^2}{2}\psi''' + iuF'',$$

$$\frac{dz}{du} = u\psi''' + F'', \qquad \frac{ds}{du} = F'',$$

* E. Salkowski, Ueber algebraish rectifizierbare Raumkurven, Math. Annalen, Vol. 67.

† L. P. Eisenhart, Fundamental parametric representation of space curves, Annals of Mathematics, Vol. 13.

and integrate, we get

$$x = \frac{1 - u^2}{2}\psi'' + u\psi' - \psi - uF' + F,$$

(3)
$$y = i\left(\frac{1 + u^2}{2}\psi'' - u\psi' + \psi + uF' - F\right),$$

$$z = u\psi'' - \psi' + F', \qquad s = F'.$$

These expressions on account of (2) give the solution of the equation (1). Now x, y, z are the cartesian coördinates of a curve c and s is its arc.

The above solutions can be put in a different form by adding and subtracting

$$\frac{1 - u^2}{2}F'', \qquad i\frac{1 + u^2}{2}F'', \qquad \text{and} \qquad uF''$$

to x, y, z respectively. Then if we put $\psi - F = \varphi$, the formulæ (3) become

$$x = \frac{1 - u^2}{2}\varphi'' + u\varphi' - \varphi + \frac{1 - u^2}{2}F'',$$

(4)
$$y = i\left(\frac{1 + u^2}{2}\varphi'' - u\varphi' + \varphi + \frac{1 + u^2}{2}F''\right),$$

$$z = u\varphi'' - \varphi' + uF'', \qquad s = F'.$$

The expressions (4) may be written

$$x = x_m + \xi, \qquad y = y_m + \eta, \qquad z = z_m + \zeta,$$

where x_m, y_m, z_m are the coördinates of the points of a minimal curve Γ, since

$$x_m = \frac{1 - u^2}{2}\varphi'' + u\varphi' - \varphi,\ y_m = i\left(\frac{1 + u^2}{2}\varphi'' - u\varphi' + \varphi\right),\ z_m = u\varphi'' - \varphi',$$

and ξ, η, ζ, containing F'', namely

$$\xi = \frac{1 - u^2}{2}F'', \qquad \eta = i\frac{1 + u^2}{2}F'', \qquad \zeta = uF'',$$

are the coördinates of the points of a curve lying on a sphere of zero radius for $\xi^2 + \eta^2 + \zeta^2 = 0$.

Since $\frac{dx_m}{du}$, $\frac{dy_m}{du}$, $\frac{dz_m}{du}$ are proportional to ξ, η, ζ respectively, the points of C lie on the corresponding tangents to Γ.

2. **Generality of Equations** (3). The coördinates of any curve may be

written in the form (4). Suppose that a curve is given by the following equations

$$x = f_1(v), \qquad y = f_2(v), \qquad z = f_3(v),$$

where f_1, f_2, f_3 are analytic functions of v in a certain domain. Form

$$ds = [(f_1'(v))^2 + (f_2'(v))^2 + (f_3'(v))^2]^{1/2}dv.$$

From (4) we have

$$\begin{aligned}
dx &= \left(\frac{1-u^2}{2}\varphi''' + \frac{1-u^2}{2}F''' - uF''\right)du, \\
dy &= i\left(\frac{1+u^2}{2}\varphi''' + \frac{1+u^2}{2}F''' + uF''\right)du, \qquad (5)\\
dz &= (u\varphi''' + uF''' + F'')du, \qquad ds = F''du.
\end{aligned}$$

Hence we have

$$dx - idy = (\varphi''' + F''')du, \qquad dz - ds = u(\varphi''' + F''')du.$$

Therefore

$$u = \frac{dz - ds}{dx - idy}. \tag{6}$$

If the curve to be dealt with is a straight line then dx/ds, dy/ds, dz/ds are constants and therefore u is a constant. Hence a straight line cannot be represented by the formulæ (4). To discuss fully the case of u a constant we consider another parameter

$$\bar{u} = \frac{dz + ds}{dx - idy},$$

which gives a similar representation of the coördinates of a curve as given by (4) in terms of the parameter u. In fact, by making

$$dx - idy = \psi'''du, \qquad ds - dz = -2F'' - \bar{u}\psi'''du,$$

we get by the definition of $\bar{u}$ the formulæ (3) and then pass to (4) in the indicated manner. We see now that in the case of a straight line both u and $\bar{u}$ are constants and conversely, if u and $\bar{u}$ are constants the curve under consideration is a straight line. But if u is a constant, $\bar{u}$ may be a variable and conversely. For the relation between u and $\bar{u}$ can be given either of the forms

$$u + \frac{2F''}{\varphi''' + F'''} = \bar{u}, \qquad u = \bar{u} + \frac{2\bar{F}''}{\bar{\varphi}''' + \bar{F}'''}.$$

Suppose now u is a constant c, then

$$\bar{u} = c - \frac{2\bar{F}''}{\bar{\varphi}''' + \bar{F}'''}$$

and $(c - \bar{u})(\bar{\varphi}''' + \bar{F}''') = 2\bar{F}''$. Substituting into the analogues of (5) for $\bar{u}$, we get

$$dx = (\bar{\varphi}''' + \bar{F}''')\frac{(1 + c\bar{u})}{2}d\bar{u}, \qquad dy = i(\bar{\varphi}''' + \bar{F}''')\frac{(1 - c\bar{u})}{2}d\bar{u},$$

$$dz = (\bar{\varphi}''' + \bar{F}''')\frac{(c + \bar{u})}{2}d\bar{u},$$

and hence

$$\frac{1 - c^2}{2}dx + i\frac{1 + c^2}{2}dy + cdz = 0,$$

or

$$\frac{1 - c^2}{2}x + i\frac{1 + c^2}{2}y + cz = A, \tag{7}$$

where A is an arbitrary constant. The equation (7) shows that the curve (x, y, z) lies in an isotropic plane in case either u or $\bar{u}$ is a constant, and can be represented by equations of the form (4) either in terms of u or $\bar{u}$. So that only in the case of a straight line the representation (4) fails. From (4) we see that

$$\frac{1 - u^2}{2}x + \frac{i(1 + u^2)}{2}y + uz = -\varphi. \tag{8}$$

So that φ, φ', φ'' and F'' are now known and we can write the expressions for x, y, z in the form (4) except in the case of a straight line.

3. **Algebraic Curves.** If the curve under consideration is an algebraic curve we have

$$x = F_1(z), \qquad y = F_2(z)$$

where F_1 and F_2 represent algebraic functions of z. Hence dx/dz, dy/dz, ds/dz are algebraic functions of z, and from (6) z is an algebraic function of u. That is x, y, z are algebraic functions of u and from (8) φ is an algebraic function of u. It follows then that in the case of an algebraic curve F'' and φ must be algebraic functions of u. The converse is also true. For if F'' and φ are algebraic functions of u, x and y are algebraic functions of z and the curve is an algebraic curve. Therefore a necessary and sufficient condition that the curves represented in the form (4) be algebraic curves is that F'' and φ be algebraic functions of u. Since $ds = F''du$, we see that if F' is an algebraic function of u, the algebraic curves corresponding to that value of s are algebraically rectifiable.

4. **The Fundamental Formulæ for the Curves Represented by (4).** We

denote the direction-cosines of the tangent, principal normal and binormal to a curve by α, β, γ; l, m, n; λ, μ, ν respectively. Then

$$\alpha = \frac{1 - u^2}{2}\chi - u, \qquad \beta = i\frac{1 + u^2}{2}\chi + iu, \qquad \gamma = u\chi + 1,$$

where

$$\chi = \frac{\varphi''' + F'''}{F''}.$$

Let us call σ the arc of the spherical indicatrix of the tangents, and σ_1 that of the binormals. Then $d\sigma = \sqrt{\chi^2 - 2\chi'}\,du$, and if ρ denotes the radius of curvature, we have

$$\rho = \frac{ds}{d\sigma} = \frac{F''}{\sqrt{\chi^2 - 2\chi'}}.$$

If we make use of Frenet's formulæ, we find

$$l = \rho\frac{d\alpha}{ds}, \qquad m = \rho\frac{d\beta}{ds}, \qquad n = \rho\frac{dx}{ds},$$

so that the direction-cosines of the principal normals become

$$l = \frac{1}{\sqrt{\chi^2 - 2\chi'}}\left\{\frac{1 - u^2}{2}\chi' - u\chi - 1\right\},$$

$$m = \frac{i}{\sqrt{\chi^2 - 2\chi'}}\left\{\frac{1 + u^2}{2}\chi' + u\chi + 1\right\},$$

$$n = \frac{1}{\sqrt{\chi^2 - 2\chi'}}\{u\chi' + \chi\}.$$

For $\lambda = \beta n - \gamma m$, etc., we obtain

$$\lambda = \frac{-i}{\sqrt{\chi^2 - 2\chi'}}\left\{\frac{1 - u^2}{2}(\chi' - \chi^2) + u\chi + 1\right\},$$

$$\mu = \frac{1}{\sqrt{\chi^2 - 2\chi'}}\left\{\frac{1 + u^2}{2}(\chi' - \chi^2) - u\chi - 1\right\},$$

$$\nu = \frac{-i}{\sqrt{\chi^2 - 2\chi'}}\{u(\chi' - \chi^2) - \chi\},$$

and

$$d\sigma_1 = i\cdot\frac{\chi^3 - 3\chi\chi' + \chi''}{\chi^2 - 2\chi'}du.$$

If τ denotes the radius of torsion, then

$$\tau = \frac{ds}{d\sigma_1} = -i\frac{(\chi^2 - 2\chi')F''}{\chi^3 - 3\chi\chi' + \chi''}, \qquad \frac{\rho}{\tau} = i\frac{\chi^3 - 3\chi'\chi + \chi''}{(\chi^2 - 2\chi')^{3/2}}.$$

Since

$$d\sigma_1 = i\left(\chi - \frac{\chi\chi' - \chi''}{\chi^2 - 2\chi'}\right)du$$

we have

$$\sigma_1 = i\left(\int \chi du - \log \sqrt{\chi^2 - 2\chi'}\right).$$

Therefore if χ is the derivative of a function $M(u)$, σ_1 is expressible without any sign of integration. We shall see that this is the case for all spherical curves.

5. **Plane Curves.** By using formulæ (3) we obtain the coördinates of a plane curve in terms of the parameter u. For the sake of simplicity we suppose that the plane of the curve is the xy-plane. Hence, if we take $z = 0$, we find that to within an additive constant

$$F = -u\psi' + 2\psi.$$

Substituting the values of F and F' in (3), we get

$$x = \frac{1 + u^2}{2}\psi'' - u\psi' + \psi, \qquad y = i\left(\frac{1 - u^2}{2}\psi'' + u\psi' - \psi\right),$$

$$s = F' = \psi' - u\psi''.$$

The formulæ for α, β, γ are quite interesting for the plane curves in terms of the parameter u. They are

$$\alpha = -\frac{1 + u^2}{2u}, \qquad \beta = -i\frac{1 - u^2}{2u}, \qquad \gamma = 0.$$

Since $dz/ds = u\chi + 1$, in the case of plane curves $\chi = -1/u$. So that for all plane curves the ratio $(\varphi''' + F''')/F''$ has the same value, $-1/u$. Another interesting fact is that the slopes of the tangents to the plane curves are given by the formula

$$i\frac{1 - u^2}{1 + u^2},$$

and are independent of the function ψ. We find also for the plane curves

$$\rho = iu^2\psi''',$$

and for the tangent lines the equation

$$i(1 + u^2)y + (1 - u^2)x + 2(u\psi' - \psi) = 0$$

6. **Spherical Curves.** To deduce the equations for spherical curves we assume that the center of the sphere of radius unity is at the origin so that the coördinates x, y, z of the spherical curve satisfy the equation

$$x^2 + y^2 + z^2 = 1,$$

which for the expressions (4) necessitates the relation

$$\varphi'^2 - 2\varphi\varphi'' - 2\varphi F'' = 1.$$

Hence

$$F'' = \frac{\varphi'^2 - 1 - 2\varphi\varphi''}{2\varphi},$$

and the equations (4) for a spherical curve become

$$x = \frac{1 - u^2}{2}\varphi'' + u\varphi' - \varphi + \frac{1 - u^2}{2} \cdot \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi},$$

$$y = i\left(\frac{1 + u^2}{2}\varphi'' - u\varphi' + \varphi\right) + i\frac{1 + u^2}{2} \cdot \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi},$$

$$z = u\varphi'' - \varphi' + u\frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi},$$

$$ds = \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi}du.$$

The function designated by χ we obtain by computing $(\varphi''' + F''')/F''$ and find

$$\chi = \frac{-\varphi'}{\varphi} = \frac{-d\log\varphi}{du}.$$

Substituting this value of χ into the formulæ of § 4 we obtain the corresponding formulæ for spherical curves. We call attention to the formula for $d\sigma_1$, which after integration is reduced to

$$\sigma_1 = i\log\frac{1}{\varphi} \cdot \frac{d\sigma}{du}.$$

A GENERAL METHOD OF SUMMATION OF DIVERGENT SERIES.

By Lloyd L. Smail.

Introduction.

In articles by Hardy and Chapman in the Quarterly Journal of Mathematics, vol. 42 (1911), and by Chapman in the Quarterly Journal, vol. 43 (1912), a general definition of summability of divergent series is given and its general properties discussed. Silverman also, in his dissertation "On the Definition of the Sum of a Divergent Series," discusses certain general definitions of summability. In the present paper, a similar general definition is given, but in such a form as to include as special cases most of the known particular methods, namely, the definitions of Cesàro, Hölder, Riesz, de la Vallée-Poussin, Plancherel, LeRoy, Borel's integral definition, Borel's exponential and generalized exponential methods, and the so-called Euler power series method. Certain general properties of this method are developed: the summability of convergent and properly divergent series, and the continuity, integration, and differentiation of uniformly summable series. Applications of these general results are then made to the various particular methods included as special cases under this general method.

The General Method.

Let f_i be a function of the variables n and x, defined for all positive integral values of i, and for all positive values of n and x. Let $\sum\limits_0^\infty a_n$ be any given series, convergent or divergent.

If the expression $\sum\limits_{i=0}^n a_i f_i$ has a repeated double limit S:

$$\text{(I)} \qquad \underset{x}{\mathrm{L}}\ \underset{n}{\mathrm{L}} \sum_{i=0}^{n} a_i f_i(n, x) = S,$$

where $\underset{n}{\mathrm{L}}$ stands for $\lim\limits_{n\to\infty}$, we shall say that the series Σa_n is *summable* (A_f), by the summation-function $f_i(n, x)$, and that S is its *Sum* (or generalized sum).

In the above double limit, it is meant that n shall $\to \infty$ first, while x is held fixed, and then $x \to \infty$.

We shall say that a series of functions $\sum_0^\infty a_n(u)$ is *uniformly summable* (A_f) when the limit (I) is approached uniformly with respect to u in some interval (a, b); that is, when the simple limit $\mathbf{L}_n \sum_{i=0}^n a_i f_i(n, x) = S_x(u)$ say, is approached uniformly with respect to u in (a, b) and uniformly with respect to x for large values of x, and then the limit as $x \to \infty$: $\mathbf{L}_n S_x(u) = S(u)$, is approached uniformly with respect to u, according to the usual definition of uniform approach to a simple limit.

Theorem I. If two series Σa_n and Σb_n are summable (A_f) with Sums A and B, then the series $\Sigma(a_n \pm b_n)$ is summable (A_f) with the sum $A \pm B$.

The proof of this proposition is obvious.

Theorem II. If $\Sigma a_n(u)$ is uniformly summable (A_f) with respect to u in an interval (a, b), and if the terms $a_n(u)$ are continuous functions of u in (a, b), then its Sum $S(u)$ is a continuous function of u in (a, b).

This proposition is a direct consequence of the continuity of a uniform limit of continuous functions.

Theorem III. If $\sum_0^\infty a_n(u)$ is uniformly summable (A_f) in an interval (a, b) with Sum $S(u)$, and if the terms $a_n(u)$ are integrable in this interval, then the series obtained by integrating the given series term-by-term: $\sum_0^\infty \int_{c_1}^{c_2} a_n(u)du$, is summable (A_f), with Sum equal to $\int_{c_1}^{c_2} S(u)du$, where (c_1, c_2) is contained in (a, b).

If $\mathbf{L}_n^*$ means a uniform limit as $n \to \infty$, the theorem on integration of uniformly convergent series shows that the process $\mathbf{L}_n^*$ followed by the process $\int_{c_1}^{c_2}$ yields the same result as the process $\int_{c_1}^{c_2}$ followed by the process $\mathbf{L}_n^*$.

By hypothesis,

$$(a) \qquad S(u) = \mathbf{L}_x^*\left(\mathbf{L}_n^* \sum_{i=0}^n a_i(u)\cdot f_i(n, x)\right).$$

Then

$$(b) \qquad \int_{c_1}^{c_2} S(u)du = \int_{c_1}^{c_2} \mathbf{L}_x^*\left(\mathbf{L}_n^* \sum_{i=0}^n a_i(u)\cdot f_i(n, x)\right)du.$$

Remembering that $\int_{c_1}^{c_2}$ and $\sum_{i=0}^n$ are permutable processes, we obtain from (b),

$$\text{(c)} \qquad \int_{c_1}^{c_2} S(u)du = \mathbf{L}_x\left(\mathbf{L}_n \sum_{i=0}^{n} f_i(n, x) \cdot \int_{c_1}^{c_2} a_i(u)du\right),$$

which proves our theorem.*

Theorem IV. If $\Sigma a_n(u)$ is summable (A_f) with Sum $S(u)$, and if the terms $a_n(u)$ are differentiable, and if the series $\Sigma a_n'(u)$ obtained from $\Sigma a_n(u)$ by term-by-term differentiation, is uniformly summable (A_f) with respect to u in (a, b), with Sum $\sigma(u)$, and if the terms $a_n'(u)$ are integrable, then

$$\sigma(u) = \frac{d}{du} S(u).$$

This theorem may be proved by applying the theorem on integration to the series $\Sigma a_n'(u)$ in the same way as for the analogous case of uniformly convergent series.

So far, no restrictions whatever have been imposed upon the summation-function $f_i(n, x)$. In order to make summability (A_f) an actual generalization of convergency, we subject the function $f_i(n, x)$ to certain general restrictive conditions, and set up the following definition.

Definition. If $f_i(n, x)$ satisfies the following conditions:

(II) $\begin{cases} 1°. \text{ When } n \text{ and } x \text{ are fixed, the sequence } (f_i) \text{ is positive and decreasing;} \\ 2°. \ \mathbf{L}_x \mathbf{L}_n f_i(n, x) = 1 \text{ for } i \text{ fixed;} \end{cases}$

and if limit (I) exists:

$$\mathbf{L}_x \mathbf{L}_n \sum_{0}^{n} a_i f_i(n, x) = S,$$

then we shall say that the series Σa_n is *summable* (A_f') with *Sum* S.

We then have the two following fundamental results:

Theorem V. If Σa_n is convergent with sum S, it is also summable (A_f') with Sum S.

For, let ϵ be any arbitrarily small positive number < 1, then since the given series is convergent, we can determine m so large that

$$\text{(a)} \qquad \left|\sum_{i=m}^{m+p} a_i\right| < \frac{\epsilon}{4}, \qquad (p = 1, 2, 3, \cdots);$$

let m now remain fixed, and then by 2°, integers X and N_X (depending on X) can be determined so large that

* This proof can be carried out using a weaker uniformity than that demanded in the definition given for uniform summability; for it is evidently sufficient to demand the uniformity of $\mathbf{L}_n \sum_{0}^{n} a_i(u) f_i(n, x)$ for every fixed x *separately* instead of requiring it for all x's simultaneously, and then, as before, the uniformity of $\mathbf{L}_x(\ )$.

(b) $|f_i(n, x) - 1| < \frac{\epsilon}{4A}$, $\left\{i = 0, 1, 2, \cdots, m-1;\ A \equiv \sum_0^{m-1} |a_i|\right\}$,

for every $x > X$ and $n > N_X$, and

(c) $$|f_m(n, x) - 1| < \frac{\epsilon}{4}$$

for every $x > X$ and $n > N_X$.

Write

$$\left|\sum_0^n a_i f_i(n, x) - S\right| \leqq \left|\sum_0^{m-1} a_i f_i - \sum_0^{m-1} a_i\right| + \left|\sum_m^\infty a_i\right| + \left|\sum_m^n a_i f_i\right|.$$

By (b) we have

$$\left|\sum_0^{m-1} a_i f_i - \sum_0^{m-1} a_i\right| < \frac{\epsilon}{4},$$

by (a),

$$\left|\sum_m^\infty a_i\right| < \frac{\epsilon}{4},$$

and by Abel's lemma, using 1° and (c),

$$\left|\sum_m^n a_i f_i\right| < |f_m(n, x)| \cdot \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

$$\therefore \left|\sum_0^n a_i f_i - S\right| < \epsilon$$

for every $x > X$ and $n > N_X$, hence it follows easily that

$$\mathop{\mathrm{L}}_x \mathop{\mathrm{L}}_n \sum_0^n a_i f_i = S.$$

Theorem VI. A properly divergent series is not summable (A_f') with finite Sum; that is, if $\mathop{\mathrm{L}}_n \sum_0^n a_i = +\infty$, then $\mathop{\mathrm{L}}_x \mathop{\mathrm{L}}_n a_i f_i = +\infty$.

If we put $\sum_{p=0}^i a_p \equiv s_i$, we have identically

(a) $$\sum_{i=0}^n a_i f_i = \sum_{i=0}^{n-1} s_i(f_i - f_{i+1}) + s_n f_n.$$

Let K be any arbitrarily large positive number, then we may write

$$s_i = K + r_i \text{ for } i > m, \text{ where } r_i > 0.$$

Substituting this in (a), we get

$$
\begin{aligned}
\sum_{i=0}^{n} a_i f_i &= \sum_{i=0}^{m-1} (s_i - K)(f_i - f_{i+1}) + K \sum_{i=0}^{n-1} (f_i - f_{i+1}) \\
(b) \qquad &\qquad + \sum_{i=m}^{n-1} r_i(f_i - f_{i+1}) + s_n f_n \\
&= \sum_{i=0}^{m-1} (s_i - K)(f_i - f_{i+1}) + K \cdot f_0 + \sum_{i=m}^{n-1} r_i(f_i - f_{i+1}) + r_n f_n.
\end{aligned}
$$

Now keeping m fixed, and passing to the limit $\underset{x}{\mathbf{L}}\,\underset{n}{\mathbf{L}}$, applying the conditions 1°, 2°, and noting that the last two terms are always positive, we get

$$\underset{x}{\mathbf{L}}\,\underset{n}{\mathbf{L}} \sum_{0}^{n} a_i f_i > K,$$

and since K can be made as large as we please, it follows that

$$\underset{x}{\mathbf{L}}\,\underset{n}{\mathbf{L}} \sum_{0}^{n} a_i f_i = +\infty.$$

Particular Methods.

For the various particular methods included as special cases of our general definition of summability, we have the following summation-functions:

*Cesàro's Method:**

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(k+n)\cdots(k+n-i+1)} \; (i = 1, 2, 3, \cdots, n), \qquad f_0(n, x) = 1,$$

where k is any real number except a negative integer.

Hölder's Method:†

$$f_i(n, x) = \left(1 - \frac{i}{n+1}\right)^k \qquad (i = 0, 1, 2, \cdots, n),$$

where k is any real number.

Riesz's Method:‡

$$f_i(n, x) = \left\{1 - \frac{\lambda(i)}{\lambda(n)}\right\}^k (i = 1, 2, 3, \cdots, n), \qquad f_0(n, x) = 1,$$

where $\lambda(n)$ is a positive monotonic function of n, increasing to ∞ with n.

Vallée-Poussin's Method:||

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(n+1)(n+2)\cdots(n+i)} \; (i = 1, 2, \cdots, n), \qquad f_0(n, x) = 1.$$

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* Cesàro, Bulletin des sciences math., sér. 2, vol. 14, pp. 114–120; Chapman, Proc. London Math. Soc., ser. 2, vol. 9, pp. 369–409; Knopp, Sitzungsberichte der Berliner Math. Ges., vol. 7, pp. 1–12.

† Hölder, Math. Annalen, vol. 20, pp. 535–549.

‡ Riesz, Paris Comptes Rendus, vol. 148, pp. 1658; vol. 149, pp. 18–21, 909–912.

|| Vallée-Poussin, Bull. de la Classe des Sciences de l'Académie Royale de Belgique (1908), pp. 193–254.

*Plancherel's Method:**

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(n+2)(n+3)\cdots(n+i+1)} \quad (i = 1, 2, \cdots, n), \qquad f_0(n, x) = 1.$$

Leroy's Method:†

$$f_i(n, x) = \frac{\Gamma(ie^{-1/x} + 1)}{\Gamma(i + 1)}.$$

Euler's Power Series Method:

$$f_i(n, x) = e^{-i/x}.$$

Borel's Integral Definition:‡

$$f_i(n, x) = \int_0^\infty e^{-t} \cdot \frac{t^i}{i!} dt.$$

Borel's Exponential Definition:§

$$f_i(n, x) = e^{-x}\{E_n(x) - E_{i-1}(x)\},$$

where

$$E_n(x) \equiv \sum_{i=0}^{n} \frac{x^i}{i!}.$$

Borel's Generalized Exponential Definition:||

$$f_i(n, x) = e^{-x^k}\{E_n(x^k) - E_{i-1}(x^k)\}.$$

It can be shown without much difficulty that all of these special summation functions, corresponding to the various particular methods discussed above, satisfy the conditions (II) imposed on the function $f_i(n, x)$ for summability (A_f'), and hence all of these special methods appear as special cases of our general definitions of summability (A_f) and (A_f'). If we now apply our general theorems I–VI to these various particular definitions of summability, we get a number of results, some new, and some already well-known, but the important point here is that they all follow at once from a few general propositions.

Thus, all convergent series are summable by all of the particular methods enumerated above, with a generalized Sum equal to the ordinary sum, and properly divergent series are not summable with finite Sum by any of these methods; and if a series of functions is uniformly summable by any of these methods, it has properties with respect to continuity, integration, and differentiation analogous to those for uniformly convergent series.

* Plancherel, Rend. Circ. Mat. di Palermo, vol. 33, pp. 41–66.
† LeRoy, Annales de Toulouse, sér. 2, vol. 2, pp. 317–430.
‡ Borel, Leçons sur les séries divergentes, p. 98.
§ Borel, Leçons sur les séries divergentes, p. 97.
|| Borel, Leçons sur les séries divergentes, p. 129.

UNIVERSITY OF WASHINGTON,
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ON QUATERNIONS AND THEIR GENERALIZATION AND THE HISTORY OF THE EIGHT SQUARE THEOREM.

By L. E. Dickson.

1. **Objects of the paper.** We shall present the history of the generalizations to four and eight squares of the familiar formula

$$(1)\quad (a^2 + b^2)(\alpha^2 + \beta^2) = r^2 + s^2, \qquad r = a\alpha - b\beta, \qquad s = a\beta + b\alpha,$$

and an elementary exposition of Hurwitz's proof that such a formula holds only for 2, 4 or 8 squares. For these three cases we shall show that the formula admits of a simple interpretation concerning the norms of numbers which are ordinary complex numbers, quaternions or numbers of Cayley's algebra with 8 units. No knowledge of quaternions or the latter algebra will be presupposed, but their more fundamental algebraic properties will be developed in detail.

A clear exposition will first be given (§§ 1–5) of the main results of our subject. This will be followed (§§ 6–28) by an account of its history, which is believed to omit no paper on the eight square theorem and its generalization

2. **Ordinary complex numbers.** Let a and b be any real numbers. Then the complex number $a + bi$ is said to have the *norm* $a^2 + b^2$. Formula (1) evidently expresses the property that the norm of the product $r + si$ of the complex numbers $a + bi$ and $\alpha + \beta i$ equals the product of their norms.

To prepare the way for our introduction to quaternions and Cayley's algebra, we shall present briefly W. R. Hamilton's definition of complex numbers by means of couples of real numbers. Two couples (a, b) and (α, β) are called equal if and only if $a = \alpha$, $b = \beta$. Addition and multiplication are defined by

$$(a, b) + (\alpha, \beta) = (a + \alpha, b + \beta), \qquad (a, b)(\alpha, \beta) = (r, s),$$

where r and s are given by (1). If m is any real number, we define $m(a, b)$ and $(a, b)m$ to be (ma, mb). Writing 1 for $(1, 0)$ and i for $(0, 1)$, we have

$$(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1) = a + bi.$$

The previous definition of addition and multiplication of couples gives

$$(a + bi) + (\alpha + \beta i) = a + \alpha + (b + \beta)i, \qquad (a + bi)(\alpha + \beta i) = r + si.$$

3. Quaternions. Consider quadruples (a, b, c, d) of real or complex numbers a, b, c, d. Define addition and multiplication by

$$(a, b, c, d) + (\alpha, \beta, \gamma, \delta) = (a + \alpha, b + \beta, c + \gamma, d + \delta),$$

$$(a, b, c, d) \times (\alpha, \beta, \gamma, \delta) = (A, B, C, D),$$

where

$$\text{(2)} \qquad \begin{cases} A = a\alpha - b\beta - c\gamma - d\delta, & B = a\beta + b\alpha + c\delta - d\gamma, \\ C = a\gamma - b\delta + c\alpha + d\beta, & D = a\delta + b\gamma - c\beta + d\alpha. \end{cases}$$

No attempt will be made here to explain why we select these values for $A, \cdots, D$; it is not our purpose to explain how quaternions were discovered or how they may be made to enter naturally,* as we aim merely to give a logical basis for quaternions. Consider the four particular quadruples

$$1 = (1, 0, 0, 0), \quad i = (0, 1, 0, 0), \quad j = (0, 0, 1, 0), \quad k = (0, 0, 0, 1),$$

called the *units*. Define $m(a, b, c, d)$ or $(a, b, c, d)m$ to be (ma, mb, mc, md), where m is any complex number. Then

$$(a, b, c, d) = (a, 0, 0, 0) + \cdots + (0, 0, 0, d) = a + bi + cj + dk,$$

$$\text{(3)} \qquad \begin{matrix} i^2 = j^2 = k^2 = -1, & ij = k, & ji = -k, & & \\ & jk = i, & kj = -i, & ki = j, & ik = -j. \end{matrix}$$

Henceforth we discard the quadruple notation and employ

$$q = a + bi + cj + dk, \qquad Q = \alpha + \beta i + \gamma j + \delta k,$$

called quaternions. In view of our earlier definitions, their sum is $a + \alpha + (b + \beta)i + \cdots$ and their product is $A + Bi + Cj + Dk$, where $A, \cdots, D$ have the values (2). This product may be found by performing the multiplication as in formal algebra, care being taken not to permute two factors i, j, k, and then simplifying the result by use of (3). For example, $(i + 2j)(j + k) = k - j - 2 + 2i$. Note that, while multiplication is not commutative, it is associative since $(ij)k = -1 = i(jk)$, etc.

The quaternion $q' = a - bi - cj - dk$ is called the *conjugate* to q. We readily verify that $qq' = q'q = a^2 + b^2 + c^2 + d^2$, which is called the *norm* $N(q)$ of q. For the moment, let a, b, c, d be real numbers, so that q is a real quaternion; if $q \neq 0$, then $N(q) \neq 0$, and q has the inverse $q^{-1} = q'/N(q)$. Thus, if $q \neq 0$, $qQ = q_1$ has the unique solution $Q = q^{-1}q_1$, and $Qq = q_1$ has the unique solution $Q = q_1q^{-1}$, so that both right-hand

* This topic is presented in an elementary manner in Dickson's Linear Algebras, Cambridge University Tract No. 16, pp. 9–12, and from another standpoint in his article "On the relation between linear algebras and continuous groups," Bull. Amer. Math. Soc., 22, 1915, 53–61.

and left-hand division are always uniquely possible if the divisor is a real quaternion not zero.

The conjugate of qQ equals the product $Q'q'$ of the conjugates of the factors taken in reverse order, as shown by interchanging the Roman and Greek letters in the sums (2) and afterwards changing the signs of $b, c, d, \beta, \gamma, \delta$.

The norm of qQ is $qQ \cdot Q'q'$ by definition. By the associative law, this may be written $q(QQ')q'$. Since QQ' is an ordinary number, it is commutative with q' in view of our earlier definition of $m(a, b, c, d)$ and $(a, b, c, d)m$. The result is now the product of the norms qq' and QQ' of q and Q. Hence the norm of a product of two quaternions equals the product of their norms, *i. e.*,

$$\begin{aligned}(4)\qquad &(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)\\ &\qquad = A^2 + B^2 + C^2 + D^2 \qquad (A, \cdots, D \text{ as in } (2)).\end{aligned}$$

Much earlier than Hamilton's invention of quaternions in 1843, Euler* discovered formula (4) while investigating the elegant theorem that every positive integer is a sum of four integral squares, the theorem following from (4) if proved for every prime number; he also used (4) in his later paper on orthogonal substitutions.

4. **Cayley's algebra.** A. Cayley† defined an algebra with the 8 units $1, i_1, \cdots, i_7$, such that $i_1^2 = -1, \cdots, i_7^2 = -1$,

$$i_1 i_2 = i_3 = -i_2 i_1, \quad i_2 i_3 = i_1 = -i_3 i_2, \quad i_3 i_1 = i_2 = -i_1 i_3,$$

and six similar sets of six relations with 1, 2, 3 replaced by 1, 4, 5; 6, 2, 4; 6, 5, 3; 7, 2, 5; 7, 3, 4; 1, 7, 6; respectively. Then

$$(x_0 + x_1 i_1 + \cdots + x_7 i_7)(x_0' + x_1' i_1 + \cdots + x_7' i_7) = A_0 + A_1 i_1 + \cdots + A_7 i_7,$$

where, if we employ the abbreviations $jk = x_j x_k' - x_k x_j'$, $\overline{0j} = x_0 x_j' + x_j x_0'$,

$$\begin{aligned} A_0 &= x_0 x_0' - x_1 x_1' - \cdots - x_7 x_7', & A_1 &= 23 + 45 + 76 + \overline{01},\\ A_2 &= 31 + 46 + 57 + \overline{02}, & A_3 &= 12 + 65 + 47 + \overline{03},\\ A_4 &= 51 + 62 + 73 + \overline{04}, & A_5 &= 14 + 36 + 72 + \overline{05},\\ A_6 &= 24 + 53 + 17 + \overline{06}, & A_7 &= 25 + 34 + 61 + \overline{07}.\end{aligned}$$

He called Σx_i^2 the modulus (norm) of $x_0 + \cdots + x_7 i_7$ and stated that the

* Corresp. Math. Phys. (ed., P. H. Fuss), I, 1843, 452, letter to Goldbach, May 4, 1748. Novi Comm. Acad. Petrop., 5, 1754–5, 3; 15, 1770, 75; Comm. Arith. Coll., I, 230, 427.

† Phil. Mag., London, (3), 26, 1845, 210 [30, 1847, 257–8]; Coll. Math. Papers, I, 127 [301]. In A_4 his 87 is a misprint for 73.

norm of a product equals the product of the norms of the two factors:

$$\left(\sum_{i=0}^{7} x_i^2\right)\left(\sum_{i=0}^{7} x_i'^2\right) = \sum_{i=0}^{7} A_i^2. \tag{5}$$

The last result, as well as another important property of the algebra, can be proved without computation by representing the algebra as a quasi-binary algebra.* Since $1, i_1, i_2, i_3$ satisfy the relations (3) for the quaternion units, we may replace them by $1, i, j, k$. Then the remaining four units are $e = i_4$, $ie = i_5$, $je = i_6$, $ke = i_7$. Hence every number of the algebra is of the form $q + Qe$, where q and Q are linear functions of $1, i, j, k$ and hence are quaternions. It can be verified that Cayley's 49 relations, giving the product of two equal or distinct units $i_1, \cdots, i_7$, are together equivalent to the single formula

$$(q + Qe)(r + Re) = qr - R'Q + (Rq + Qr')e, \tag{6}$$

where r' and R' are the quaternions conjugate to r and R. The reader need not verify the equivalence stated, but may take (6) as the rule of multiplication for the numbers of the algebra to be considered henceforth, since Cayley's algebra has been introduced here merely for historical background and will not be further employed in his form.

Define the norm of $q + Qe$ to be $qq' + QQ'$, which is a sum of 8 squares. Taking $r = q'$, $R = -Q$, in (6), we get

$$(q + Qe)(q' - Qe) = qq' + QQ',$$

so that the norm of $q + Qe$ is its product by its *conjugate* $q' - Qe$. Since multiplication does not here obey the associative law, we cannot conclude at once, as we did for quaternions in §3, that the norm of a product equals the product of the norms of the two factors. However, we obtain a short proof by use of a device. Express the right member of (6) in the form $t + Te$ by setting

$$t = qr - R'Q, \qquad T = Rq + Qr'. \tag{7}$$

Its norm $tt' + TT'$ is seen, by direct multiplication and use of the fact that the norm of qr is $r'q'$, to equal $\alpha - \beta + \gamma$, where

$$\alpha = RqrQ' + Qr'q'R', \qquad \beta = qrQ'R + R'Qr'q',$$

$$\gamma = qrr'q' + R'QQ'R + Rqq'R' + Qr'rQ' = (qq' + QQ')(rr' + RR').$$

The last equality is a consequence of the fact that rr' is an ordinary number and hence can be interchanged with q', etc. Our device occurs in the proof that $\alpha = \beta$. Note that the conjugate of the first term of α

* Dickson, Trans. Amer. Math. Soc., 13, 1912, 72; Linear Algebras, 1914, 15.

equals the second term of α, so that α is an ordinary number and hence is commutative with every quaternion. Hence $\alpha = R'\alpha R \div RR'$, which is seen to equal β. In the excluded case $R = 0$, evidently $\alpha = \beta = 0$. Hence the norm of the product (6) equals the product of the norms of the factors. Thus we can write down an 8 square formula of type (5).

Moreover, both right-hand and left-hand division except by zero is always possible and unique in our algebra composed of the numbers $q + Qe$, provided we restrict q and Q to be real quaternions. Of the two types of division consider that in which the second factor $r + Re$ and the product $t + Te$ are given, while the first factor $q + Qe$ is to be found. Thus we seek to solve equations (7) for q and Q. Multiply the second equation (7) by r on the right and replace qr by its value from the first equation; we get

$$(rr' + RR')Q = Tr - Rt.$$

Again, multiply the first equation by r' on the right and eliminate Qr'; thus

$$(rr' + RR')q = tr' + R'T.$$

Since $rr' + RR'$ equals the sum of the squares of eight real numbers, it is zero if and only if $r = R = 0$. Similarly, equations (7) can be solved for $r + Re$ unless $q = Q = 0$.

We have now accomplished one of the aims of the paper, having exhibited linear algebras in 2, 4 and 8 units for which the norm of a product equals the product of the norms of the factors (thus giving the 2, 4 and 8 square theorems), and such that, if the coördinates of the numbers of the algebra be restricted to be real numbers, both right-hand and left-hand division except by zero are possible and unique. While the three algebras have in common these two fundamental properties, they differ in other respects. For complex numbers multiplication is both commutative and associative, for quaternions it is associative but not commutative, for Cayley's algebra of 8 units it is neither commutative nor associative. What additional properties must be given up to obtain a similar linear algebra in more than 8 units? We shall prove in §5 that there exists no linear algebra in more than 8 units for which the norm is a sum of squares and the norm of a product equals the product of the norms of the factors.

5. **Hurwitz's Theorem.*** We seek the values of n for which there exists

* Göttingen Nachrichten, 1898, 309–316. Since experience shows that graduate students fail to follow various steps merely outlined by Hurwitz, we shall here give the proof in detailed, amplified form. As we shall employ (a_{ij}) to denote a matrix and not a linear transformation, we must invert the order of factors in his products.

an identity (as to the x's and y's) of the form

$$(8) \qquad (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_n^2,$$

where $z_1, \cdots, z_n$ are linear in $x_1, \cdots, x_n$ and also in $y_1, \cdots, y_n$. Let

$$(9) \qquad z_i = a_{i1}y_1 + \cdots + a_{in}y_n \qquad (i = 1, \cdots, n),$$

where the a_{ij} are linear functions of $x_1, \cdots, x_n$. We employ the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix},$$

where A' is derived from A by the interchange of its rows and columns, and is called the conjugate (or transposed) of A. In case the diagonal elements a_{ii} all equal a and the elements not in the diagonal are all zero, we shall write aI for A, where I is the unit (or identity) matrix and has the property that $IB = BI = B$ for every matrix B of n rows and n columns. The quadratic form

$$\sum_{i,j=1}^{n} b_{ij}z_iz_j = b_{11}z_1^2 + 2b_{12}z_1z_2 + b_{22}z_2^2 + \cdots \qquad (b_{ij} = b_{ji})$$

is said to have the matrix $B = (b_{ij})$, whose ith row is $b_{i1}, b_{i2}, \cdots, b_{in}$. If we replace the variables $z_1, \cdots, z_n$ by the expressions (9), we evidently obtain a new quadratic form in the variables $y_1, \cdots, y_n$; its matrix is known* to equal $A'BA$. In particular, let the quadratic form be $z_1^2 + \cdots + z_n^2$, whose matrix is $B = I$; then the quadratic form derived by replacing $z_1, \cdots, z_n$ by the expressions (9) has the matrix $A'A$, a fact which can be verified at once without making use of the standard theorem just quoted. Now we desire that the resulting quadratic form in $y_1, \cdots, y_n$ shall be the left member of (8), whose matrix is aI, where $a = x_1^2 + \cdots + x_n^2$. Hence there exists an identity (8) if and only if there exist n^2 linear functions a_{ij} of $x_1, \cdots, x_n$, whose matrix (a_{ij}) is denoted by A, such that

$$(10) \qquad A'A = (x_1^2 + \cdots + x_n^2)I.$$

Since each element of matrix A is a linear function of $x_1, \cdots, x_n$, and since the sum of several matrices is a matrix whose elements are the sums of the corresponding elements in the matrices added, it follows that $A = x_1A_1 + \cdots + x_nA_n$, where $A_1, \cdots, A_n$ are matrices with constant elements. Thus in $A'A$ the coefficient of x_n^2 is $A_n'A_n$, which equals I by (10). Let $B_i = A_n'A_i$ $(i = 1, \cdots, n - 1)$, whence $A_i = A_nB_i$, A_i'

* Bôcher, Introduction to Higher Algebra, p. 129.

$= B_i'A_n'$, and $A'A$ equals

$$(x_1B_1' + \cdots + x_{n-1}B_{n-1}' + x_n)A_n' \cdot A_n(x_1B_1 + \cdots + x_{n-1}B_{n-1} + x_n).$$

Since $A_n'A_n = I$, (10) becomes

$$(11)\quad (x_1B_1' + \cdots + x_{n-1}B_{n-1}' + x_n)(x_1B_1 + \cdots + x_{n-1}B_{n-1} + x_n) = (x_1^2 + \cdots + x_n^2)I.$$

Thus $B_i'B_i = I$, $B_i' + B_i = 0$, $B_i'B_k + B_k'B_i = 0$, whence

$$(12)\quad B_i' = -B_i, \quad B_i^2 = -I, \quad B_iB_k = -B_kB_i \qquad (i, k = 1, \cdots, n-1;\ i \neq k).$$

A matrix $B = (b_{ij})$ is called symmetric if $b_{ji} = b_{ij}$, and skew symmetric if $b_{ji} = -b_{ij}$ for every i, j; thus B is symmetric if and only if $B' = B$, and skew-symmetric if and only if $B' = -B$. The latter condition implies that $b = (-)^n b$ if b is the determinant of the matrix B of n rows and n columns. Thus $b = 0$ if n is odd. By the first two equations (12), B_i is skew-symmetric and its determinant is not zero, so that n is not odd. Hence there exists no identity (8) if n is odd. In what follows, we assume that n is even.

Our next step is to prove that at least half of the matrices

$$(13)\quad I, \quad B_{i_1}, \quad B_{i_1}B_{i_2}, \quad B_{i_1}B_{i_2}B_{i_3}, \quad \cdots, \quad B_1B_2\cdots B_{n-1} \qquad (i_1 < n,\ i_1 < i_2 < n, \cdots)$$

are linearly independent. There are 2^{n-1} such products since any one product either contains B_1 or does not, $\cdots$, and either contains B_{n-1} or does not. Let $G = B_{i_1} \cdots B_{i_r}$ be one of the matrices (13); it is symmetric if $r \equiv 0$ or 3 (mod 4), and skew-symmetric if $r \equiv 1$ or 2 (mod 4), since by (12)

$$G' = B_{i_r}' \cdots B_{i_1}' = (-1)^r B_{i_r} \cdots B_{i_1} = (-1)^s G,$$

where $s = r + r - 1 + r - 2 + \cdots + 1 = r(r+1)/2$ is even if $r \equiv 0, 3$ (mod 4), but odd if $r \equiv 1, 2$ (mod 4). In particular, a product of two distinct B's is skew-symmetric.

Consider the possible linear relations (with constant coefficients not all zero) which hold between the matrices (13). Such a relation $R = 0$ is called irreducible if it is not possible to express R in the form $R = R_1 + R_2$, where $R_1 = 0$ and $R_2 = 0$ represent two linear relations holding between our matrices such that no one of these matrices (13) occurs as a term of both R_1 and R_2. In particular, an irreducible linear relation does not involve both symmetric and skew-symmetric matrices, since it could

then be written in the form $M = S$, where M is the aggregate of the symmetric matrices and S is the aggregate of the skew symmetric matrices, whence $M' = S'$, $M' = M$, $S' = -S$, giving $M = 0$, $S = 0$.

Let $R = 0$ be any irreducible linear relation between the matrices (13). By multiplying R by the product of a constant and a suitably chosen matrix (13), we get a new linear relation $\rho = 0$, one term of which is I and all the remaining terms are products of matrices (13) by constants. Thus if $4B_2B_3$ is one term of R, we use the multiplier $-\frac{1}{4}B_2B_3$. We need also to know that if we multiply the matrices (13) on the left by any one (say M) of them, the products form a permutation of those matrices each prefixed with the factor $+1$ or -1. This is evident when the multiplier is B_1, since the product will contain or lack B_1 according as the multiplicand (13) lacks or contains B_1, in view of $B_1{}^2 = -I$. If the multiplier is B_2, we first replace $B_1B_2 \cdots$ by $-B_2B_1 \cdots$ and see that the former argument applies. After proving in this manner our statement when the multiplier is any B_i, we see that it holds when the multiplier is any product of the B's. Returning to our new relation $\rho = 0$, we note that it also is irreducible, since by multiplying it by a product of a constant and a suitable matrix (13) we recover our initial relation $R = 0$, which was assumed irreducible. Hence $\rho = 0$ is an irreducible relation

$$I = \Sigma c_{i_1i_2i_3}B_{i_1i_2i_3} + \Sigma d_{i_1i_2i_3i_4}B_{i_1}B_{i_2}B_{i_3}B_{i_4} + \cdots$$

involving exclusively symmetric matrices (13), so that no term contains a single B_i or a product of only two B's. Multiply all the terms of our relation by B_i on the right; we obtain an irreducible relation which therefore involves only skew-symmetric matrices (13), one term being B_i. Since a product of four distinct B's is symmetric, we conclude that $c_{i_1i_2i_3}$ is zero if i is distinct from i_1, i_2, i_3. Since i may have any value $\leqq n-1$, we have $c = 0$ unless $3 = n - 1$. To prove that every $d = 0$, take $i = i_4$; then the coefficient of $-dB_{i_1}B_{i_2}B_{i_3}$ is zero. The method used to prove $c = 0$ applies when the number r of factors B is $\equiv 3 \pmod 4$ and $r < n - 1$, since $r + 1 \equiv 0$. The method used to prove $d = 0$ applies when $r \equiv 0 \pmod 4$, since $r - 1 \equiv 3$. Hence if our relation exists, it has the form

$$I = kB_1B_2 \cdots B_{n-1}.$$

Since each member is a symmetric matrix, $n - 1 \equiv 0$ or $3 \pmod 4$. But n is even. Hence $n \equiv 0 \pmod 4$. As in the discussion of G, below (13), the square of $B_1 \cdots B_r$ is $(-1)^sI$, where $s = r(r+1)/2$. Hence $k^2 = 1$. *Thus the 2^{n-1} matrices (13) are linearly independent if $n \equiv 2 \pmod 4$; while for $n \equiv 0 \pmod 4$ they are either linearly independent or are connected by the relations which arise from $I = \pm B_1B_2 \cdots B_{n-1}$ by multiplication by*

the various matrices (13), *but are connected by no further irreducible linear relations.*

To illustrate this result, let $n = 4$. Then the 8 matrices

$$I, \quad B_1, \quad B_2, \quad B_3, \quad B_1B_2, \quad B_1B_3, \quad B_2B_3, \quad B_1B_2B_3$$

are either linearly independent or are connected by only four irreducible linear relations;

$$I = \pm B_1B_2B_3, \quad B_1 = \mp B_2B_3, \quad B_2 = \pm B_1B_3, \quad B_3 = \mp B_1B_2.$$

The latter express $B_1B_2B_3$, B_2B_3, B_1B_3, B_1B_2 linearly in terms of I, B_1, B_2, B_3, which are therefore in all cases linearly independent.

For any n, one of the reduced products of I and $B_1 \cdots B_{n-1}$ by any matrix (13) evidently contains fewer than half of the B's and the other contains more than half of the B's. Hence if irreducible linear relations exist, they serve merely to express the latter products in terms of the former. Thus in every case, the 2^{n-2} matrices (13) which are products of at most $(n - 2)/2$ factors B are linearly independent.

But if we are given any $n^2 + 1$ matrices $(a_{ij}^{(k)})$ each with n rows and n columns, we can find numbers x_k not all zero such that

$$\sum_{k=1}^{n^2+1} x_k(a_{ij}^{(k)}) = 0,$$

i. e.,

$$\sum_{k=1}^{n^2+1} x_k a_{ij}^{(k)} = 0 \qquad (i, j = 1, \cdots, n),$$

since n^2 linear homogeneous equations in $n^2 + 1$ unknowns x_k have solutions not all zero.

Hence $2^{n-2} \leqq n^2$. This is satisfied if $n \leqq 8$, but fails if $n = 10$. But if it fails for $n = m$, it fails for $n = m + 1$, since

$$2^{m+1-2} = 2 \cdot 2^{m-2} > 2m^2 > (m + 1)^2$$

if $(m - 1)^2 > 2$, and hence if $m \geqq 3$. We have now proved that $n \leqq 8$.

The case $n = 6$ is readily excluded. Then the 2^5 matrices (13) are linearly independent. But $5 + 10 + 1$ of them are skew-symmetric (those with 1, 2 or 5 factors B). Between any 16 skew-symmetric six-rowed square matrices there exists a linear relation:

$$\sum_{k=1}^{16} x_k(b_{ij}^{(k)}) = 0; \qquad \sum_{k=1}^{16} x_k b_{ij}^{(k)} = 0 \quad (i, j = 1, \cdots, 6;\ i < j),$$

it being now necessary to examine only the 15 terms to the right of the main diagonal. But 15 linear homogeneous equations in 16 unknowns x_k have solutions not all zero.

THEOREM. *Except for $n = 1, 2, 4, 8$, there exists no identity* (8) *expressing the product* $(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$ *as a sum of the squares of n bilinear functions of* $x_1, \cdots, x_n$ *and* $y_1, \cdots, y_n$.

HISTORY OF THE SUBJECT.

6. Gauss* remarked that the four square formula (4) is expressed in a simple way by

$$(Nl + Nm)(N\lambda + N\mu) = N(l\lambda + m\mu) + N(l\mu' - m\lambda'),$$

where $l, m, \lambda, \mu, \lambda', \mu'$ are complex numbers, λ' being conjugate to λ, and μ' to μ, while Nl denotes the norm of l (§2).

7. C. F. Degen† extended Euler's formula (4) to eight squares:

$$\begin{aligned}(P^2 + Q^2 + R^2 + S^2 + T^2 + U^2 + V^2 + X^2)&\\ \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + x^2)&\\ = (Pp + Qq + Rr + Ss + Tt + Uu + Vv + Xx)^2&\\ + (Pq - Qp + Rs - Sr + Tu - Ut + Vx - Xv)^2&\\ + (Pr - Qs - Rp + Sq \mp Tv \pm Ux \pm Vt \mp Xu)^2&\\ + (Ps + Qr - Rq - Sp \pm Tx \pm Uv \mp Vu \mp Xt)^2&\\ + (Pt - Qu \pm Rv \mp Sx - Tp + Uq \mp Vr \pm Xs)^2&\\ + (Pu + Qt \mp Rx \mp Sv - Tq - Up \pm Vs \pm Xr)^2&\\ + (Pv - Qx \mp Rt \pm Su \pm Tr \mp Us - Vp + Xq)^2&\\ + (Px + Qv \pm Ru \pm St \mp Ts \mp Ur - Vq - Xp)^2.&\end{aligned}$$

He stated [erroneously as we saw in §5] that there is a like formula for 2^n squares. For the case of 16 squares he gave the literal parts of the 16 bilinear functions, but left most of the signs undetermined, saying that the only difficulty is the prolixity of the ambiguities of signs. This paper has been overlooked by all subsequent writers on the subject.

8. J. T. Graves‡ communicated to W. R. Hamilton Jan. 18, 1844 (correcting some errors in signs in the formula communicated Dec. 26, 1843), a formula which differs from Cayley's (5) only in the interchange of 6 and 7, and a second formula which becomes Cayley's on writing $x_0, \cdots, x_7$ for $a, b, \cdots, h$. Hence Graves's formulas need not be inserted

* Posthumous MS., Werke, 3, 1876, 383–4.

† Mém. Acad. Sc. St. Pétersbourg, 8, années 1817–8 (1822), 207–219. There is a misprint in the sign of his term $\pm Rt$, here corrected.

‡ Proc. Roy. Irish Acad., 3, 1845–7, 527–9; Trans. Roy. Irish Acad., 21, II, 1848, 338–341; Phil. Mag. London, (3), 26, 1845, 320.

here. At first he expected that it would be possible to give an extension to 2^n squares.

9. J. R. Young's* formula, with s, t, u, v, y, z, w, x replaced by $a_1, \cdots, a_8$, is

$$(14)\quad \left(\sum_{i=1}^{8} a_i^2\right)(\Sigma\alpha_i^2) = (\Sigma a_i\alpha_i)^2 + (12 + 34 + 56 + 78)^2 + (13 + 42 + 57 + 86)^2 + (41 + 32 + 58 + 67)^2 + (15 + 62 + 73 + 48)^2 + (16 + 25 + 38 + 47)^2 + (17 + 82 + 35 + 64)^2 + (18 + 27 + 63 + 54)^2,$$

where ij denotes $a_i\alpha_j - \alpha_i a_j$. It was admitted to be equivalent to Graves's formulas. Young† stated that a like formula holds for 2^n squares, but soon afterwards admitted that this is erroneous, saying that he was prepared to prove that the proposition does not hold beyond 8 squares.

Young‡ gave a long discussion to show that the extension to a sum S_{16} of 16 squares is impossible. He exhibited a special relation $S_{16}S_{16}' = S_{16}''$ in which the roots of 8 of the squares in S_{16} are proportional to the roots of 8 of the squares in S_{16}'. He§ noted that, for $k = 2, 4, 8$, a product of a sum of km squares by a sum of kn squares can be expressed as a sum of kmn squares.

10. Cayley|| investigated the possibility of a formula for 2^n squares by introducing $2^n - 1$ symbols $a_0, b_0, \cdots$, not assumed to be commutative, but such that $a_0^2 = b_0^2 = \cdots = -1$ and

$$b_0c_0 = \pm a_0 = -c_0b_0, \quad c_0a_0 = \pm b_0 = -a_0c_0, \quad a_0b_0 = \pm c_0 = -b_0a_0.$$

Denoting this set of six equations by $a_0b_0c_0 = \pm$, let also $a_0d_0c_0 = \pm$, etc., where the sign is not necessarily the same as before, while the system of triples contains each duad once and but once, and the signs are to be chosen at will. Then

$$(w + aa_0 + bb_0 + \cdots)(w_1 + a_1a_0 + b_1b_0 + \cdots) = w_2 + a_2a_0 + b_2b_0 + \cdots,$$

where $w_2, a_2, \cdots$ are linear and homogeneous in $w, a, \cdots$ and in $w_1, a_1, \cdots$. Assume (I) that if any two triples with a common element, $e_0a_0b_0$ and $e_0c_0d_0$, occur in the system, there occur also $f_0a_0c_0, f_0d_0b_0, g_0a_0d_0, g_0b_0c_0$;

* Proc. Roy. Irish Acad., 3, 1845–7, 526–7.

† Phil. Mag., London, (3), 30, 1847, 424–5; 31, 1847, 123.

‡ Trans. Roy. Irish Acad., 21, II, 1848, 311–338. Outline in Proc. Roy. Irish Acad., 4, 1847–50, 19–20.

§ Phil. Mag., London, (3), 34, 1849, 114.

|| Phil. Mag., London, (4), 4, 1852, 515–9; Coll. Math. Papers, II, 49–52.

(II) that for any two pairs of triples, such as $e_0a_0b_0$, $e_0c_0d_0$ and $f_0a_0c_0$, $f_0d_0b_0$, the products of the signs of the triples in the first pair is the same as that in the second pair. Then

$$(w^2 + a^2 + b^2 + \cdots)(w_1^2 + a_1^2 + b_1^2 + \cdots) = w_2^2 + a_2^2 + b_2^2 + \cdots.$$

The converse was not proved, but it was stated that conditions (I) and (II) afford a complete test for the possibility of the 2^n square theorem.

T. P. Kirkman,* to whom Cayley had communicated privately the preceding test, verified that, for 15 elements $a, b, \cdots$, triples can be chosen so that (I) is satisfied, but that (II) then involves a contradiction.

11. F. Brioschi† showed that, if n is even, the square of the determinant $A = |a_{ij}|$ of order n is a skew-symmetric determinant $L = |l_{ij}|$ of order n with the general element

$$l_{rs} = a_{r1}a_{s2} - a_{r2}a_{s1} + a_{r3}a_{s4} - a_{r4}a_{s3} + \cdots + a_{rn-1}a_{sn} - a_{rn}a_{sn-1} = -l_{sr}.$$

Similarly, the square of $C = |c_{ij}|$ is $|p_{ij}|$, where $p_{rs} = c_{r1}c_{s2} - \cdots$. Let

$$AC = |A_{ij}|, \quad A_{rs} = \sum_{j=1}^{n} a_{rj}c_{sj}, \quad |A_{ij}|^2 = |L_{ij}|, \quad L_{rs} = A_{r1}A_{s2} - \cdots.$$

If the a's and c's are such that

$$(15) \qquad l_{12} = l_{34} = \cdots = l_{n-1n} = t, \quad p_{12} = \cdots = p_{n-1n} = u,$$

while the remaining l_{ij} and p_{ij} are zero, it is proved that $L_{12} = L_{34} = \cdots = L_{n-1n} = tu$, and that the remaining L_{ij} are zero. Now let $n = 8$ and take $a_{ii} = a_{11}$, $a_{ij} = -a_{ji}$ $(i \neq j)$ except for $a_{15} = a_{51} = a_{26} = a_{62} = a_{37} = a_{73} = a_{48} = a_{84}$ and take also

$$a_{12} = a_{43} = a_{56} = a_{87}, \; a_{13} = a_{24} = a_{57} = a_{68}, \; a_{14} = a_{32} = a_{58} = a_{76},$$

$$a_{16} = a_{47} = a_{52} = a_{83}, \; a_{17} = a_{28} = a_{53} = a_{64}, \; a_{18} = a_{36} = a_{54} = a_{72}.$$

Assume like relations between the c_{ij}. It is stated erroneously that relations (15) and the analogous relations between the A_{ij} hold, so that

$$\Sigma A_{1j}^2 = tu, \quad t = \Sigma a_{1j}^2, \quad u = \Sigma c_{1j}^2 \quad (j = 1, \cdots, 8).$$

Although $l_{12} = l_{34} = l_{56} = l_{78} = \Sigma a_{1j}^2$, it was pointed out by E. Sadun‡ that

$$l_{16} = 2(a_{11}a_{15} + a_{12}a_{16} + a_{13}a_{17} + a_{14}a_{18}) \not\equiv 0,$$

so that we cannot make $t \equiv \Sigma a_{1j}^2$. In a footnote, Sadun reconstructed Brioschi's proof, and obtained (14) with 5 and 7, 6 and 8 interchanged.

* Phil. Mag., London, (3), 33, 1848, 447–459, 494–509; (3), 37, 1850, 292–301.

† Jour. für Math., 52, 1856, 133–141; Opere Mat., V, p. 511.

‡ Periodico di Mat., 14, 1899, 125–139; and pamphlet of 1896.

12. A. Lebesgue* gave an 8 square formula, communicated to him to Prouhet, which apart from signs becomes Cayley's formula (5) if we write $x_0, \cdots, x_7$ for $a, b, \cdots, h$.

13. A. Genocchi† concluded that sums of 2^n squares repeat under multiplication by an erroneous argument (false even for $n = 2$) based upon sums of two squares. The error was pointed out by Sadun (§11) and earlier by A. Puchta,‡ who interpreted the correct 8 square formula by means of regular bodies with 9 vertices in space of 8 dimensions.

14. E. Mathieu§ expressed Euler's identity (4) in the form

$$S_4S_4' = S_4'', \quad S_4 = x_0^2 + x_1^2 + x_w^2 + x_{1+w}^2, \quad w^2 + w + 1 \equiv 0 \pmod 2,$$

$$x_0'' = x_0x_0' + x_1x_1' + x_wx_w' + x_{1+w}x_{1+w}', \qquad x_1'' = x_0x_1' - x_1x_0' - x_wx_{1+w}' + x_{1+w}x_w',$$

while x_w'' and x_{1+w}'' are derived from x_1'' by the substitution (z, wz), viz., $(1, w, 1 + w)$, on the subscripts. But S_4'' is unaltered also by $(0, w, 1 + w)$. Hence of the 24 permutations on the four subscripts, 12 give one decomposition into 4 squares, and 12 give another.

For $w^3 + w + 1 \equiv 0 \pmod 2$, Cayley's formula (5) can be expressed in the form

$$S_8S_8' = S_8'', \quad S_8 = \Sigma x_j^2, \quad x_0'' = \Sigma x_jx_j',$$

$$x_1'' = x_0x_1' - x_1x_0' - x_wx_{1+w}' + x_{1+w}x_w' - x_{w^2}x_{1+w^2}' + x_{1+w^2}x_{w^2}' - x_{w+w^2}x_{1+w+w^2}' + x_{1+w+w^2}x_{w+w^2}',$$

where j ranges over the eight values $0, \cdots, 1 + w^2$ appearing in

$$s = (0)(1, w, w^2, 1 + w, w + w^2, 1 + w + w^2, 1 + w^2).$$

The remaining x_j'' are derived from x_1'' by applying this substitution s, which may be written in the form (w^z, w^{z+1}), the signs of the terms of x_j'' being determined so that the terms occurring in the above S_4'' occur with the same signs in S_8''. Now x_1'' is unaltered by (w^z, w^{2z}), while $x_0''^2, \cdots, x_7''^2$ are permuted by $(w^z, w^{1/z})$; where $1/z$ is replaced by the integer congruent to it modulo 7. Hence any symmetric function of these 8 squares is unaltered by the $3 \cdot 7 \cdot 8$ substitutions

$$(w^z, w^{z'}), \quad z' = \frac{az + b}{cz + d}, \qquad ad - bc \equiv 1, 2, 4 \pmod 7.$$

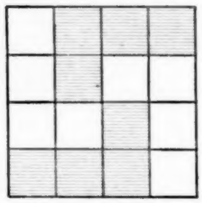
It is stated that these results cannot be extended to more than 8 squares.

* Exercices d'analyse numérique, 1859, 104; Introduction à la théorie des nombres, 1862, 65.
† Annali di Mat., 3, 1860, 202–5; Giornale di Mat., 2, 1864, 47–48.
‡ Sitzungsber. Ak. Wiss. Wien (Math.), 96, II, 1887, 110–133.
§ Jour. für Math., 60, 1862, 351–6.

15. J. J. Thomson* verified Young's formula (14) by means of relations like

$$12\cdot 34 + 13\cdot 42 + 41\cdot 32 \equiv 0. \tag{16}$$

16. E. Lucas stated† that there is a relation between the formula expressing the product of two sums of n squares as a sum of n squares for $n = 4, 8, 16$, etc., and Sylvester's‡ square diagram formed of an equal number of white cases and black cases, such that for any two lines or two columns the number of variations of colors is always equal to the number of permanences. If, in the accompanying diagram, we replace each white case by a plus sign and each black case by a minus sign, we are led to Euler's formula (4).



17. S. Roberts§ argued that a 16 square formula is impossible. He assumed in effect that an m square formula must be of the type

$$\left(\sum_{i=1}^{m} a_i^2\right)\left(\sum_{i=1}^{m} c_i^2\right) = \sum_{i=1}^{m} (a_1 c_{i1} + \cdots + a_m c_{im})^2, \tag{17}$$

where $c_{i1}, \cdots, c_{im}$ and $c_{1i}, \cdots, c_{mi}$ are permutations of $\pm c_1, \cdots, \pm c_m$, and that the formula reduces to a $\frac{1}{2}m$ square formula by setting $a_i = c_i = 0$ $(i > m/2)$. In building the m square formula from the $\frac{1}{2}m$ square formula, he made free choice between the letters not already in the scheme. He derived an unique formula for $m = 4$ and for $m = 8$, but found after a tedious examination a contradiction for $m = 16$.

18. Cayley‖ considered the linear algebra with the units $E_0 = 1, E_1, \cdots, E_7$, where $E_i^2 = -1$ $(i = 1, \cdots, 7)$ and

$$E_1E_2E_3 = \epsilon_1, \quad E_1E_4E_5 = \epsilon_2, \quad E_2E_4E_6 = \epsilon_4, \quad E_3E_4E_7 = \epsilon_6,$$

$$E_1E_6E_7 = \epsilon_3, \quad E_2E_5E_7 = \epsilon_5, \quad E_3E_5E_6 = \epsilon_7,$$

* Messenger Math., 7, 1877–8, 73–74.

† Assoc. franç. av. sc., 6, 1877, 213–4.

‡ Math. Quest. Educ. Times, 10, 1868, 74–6, 112 (diagrams for 8×8 squares and 16×16 squares). Cf. M. Jenkins, *ibid.*, 14, 1871, 22–25.

§ Quar. Jour. Math., 16, 1879, 159–170.

‖ Amer. Jour. Math., 4, 1881, 293–6; Coll. Math. Papers, XI, 368–371. Incomplete summary in Johns Hopkins University Circulars, 1882, 203.

each ϵ_i being 1 or -1, and the first symbol denotes the six equations

$$E_1E_2 = \epsilon_1E_3 = -E_2E_1, \quad E_2E_3 = \epsilon_1E_1 = -E_3E_2, \quad E_3E_1 = \epsilon_1E_2 = -E_1E_3.$$

For no values of the ϵ's is the algebra associative. We may set

$$(\Sigma a_iE_i)(\Sigma a_i'E_i) = \Sigma a_i''E_i \quad (i = 0, 1, \cdots, 7).$$

Without loss of generality we may take $\epsilon_1 = \epsilon_2 = \epsilon_3 = +1$. Then

$$(\Sigma a_i^2)(\Sigma a_i'^2) \equiv \Sigma a_i''^2,$$

if and only if $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7$. In such an algebra, $E_1E_2 \cdot E_3 = E_1 \cdot E_2E_3$ and similarly for each of the seven triads above. For the remaining 28 triads, $E_iE_j \cdot E_k = -E_i \cdot E_jE_k$. [We pass from one of these two algebras to the other by changing the signs of $E_2, \cdots, E_7$. If we take $\epsilon_4 = -1$ and change the sign of E_7, we get Cayley's earlier algebra (§4).]

19. Cayley* remarked that (16) establishes Euler's identity†

$$(\Sigma x_i^2)(\Sigma y_i^2) - (\Sigma x_iy_i)^2 = (12 + 34)^2 + (13 - 24)^2 + (14 + 23)^2.$$

The first step in forming this identity is to arrange the duads into a synthematic form: 12·34, 13·24, 14·23. The next step is to determine the signs. For 8 elements there is a single such synthematic arrangement; if 34, 56, 78 and each $1j$ are taken with positive signs, only one sign remains arbitrary, so that there are only two final schemes. For 16 elements, we have first to form 15 lines each containing the numbers $1, \cdots, 16$ in 8 duads, no duad being repeated. Only four types are found; for each it is found to be impossible to choose the signs. Cayley states that earlier writers had tacitly assumed that only one of the four types is possible and hence had not given a complete proof of the non-existence of the 16 square theorem. The question of the distinctness of the four types, apart from notation, was mentioned, but not discussed, by Cayley.

20. S. Roberts‡ remarked that Cayley's four types are all equivalent. But his directions for deriving the first from the second type are incorrect. Besides interchanging 13 with 14, and 15 with 16, and interchanging columns 13 and 14, and rows 15 and 16, it is necessary to interchange also rows 13 and 14, and columns 15 and 16. He indicated how his own process can be used to produce the four (equivalent) types.

21. F. Studnicka§ employed the product of two determinants:

$$\begin{vmatrix} a & b \\ -b' & a' \end{vmatrix} \cdot \begin{vmatrix} x & y \\ -y' & x' \end{vmatrix} = \begin{vmatrix} ax + by & -ay' + bx' \\ a'y - b'x & a'x' + b'y' \end{vmatrix}.$$

* Quar. Jour. Math., 17, 1881, 258–276; Coll. Math. Papers, XI, 294–313.

† Quoted in Math. Quest. Educ. Times, 75, 1901, 40.

‡ Quar. Jour. Math., 17, 1881, 276–280.

§ Sitzungsberichte K. Böhm Gesell. Wiss. Prag, 1883, 475–481.

Taking $a' = a, \cdots, y' = y$, we get (1). Next, let a' be the conjugate to the complex number $a, \cdots, y'$ the conjugate to y; we get Euler's (4). But he erred in employing the same formula when a' and a are conjugate quaternions, $\cdots, y'$ and y conjugate quaternions, to deduce the 8 square theorem, since he overlooked the fact that the initial formula holds only when multiplication is commutative [Vahlen, §27].

22. X. Antomari* wrote $(\alpha_i\beta_j)$ for $\alpha_i\beta_j - \beta_i\alpha_j$ and employed the identity

$$D = (\Sigma a_ix_i)(\Sigma b_iy_i) - (\Sigma a_iy_i)(\Sigma b_ix_i) = \Sigma(a_ib_j)(x_iy_j) \quad (i, j = 1, \cdots, 4; j > i).$$

In view of (16), written in a, b and again in x, y, we get

$$D = \{(a_1b_2) + (x_3y_4)\}\{(x_1y_2) + (a_3b_4)\}$$
$$+ \{(a_2b_3) + (x_1y_4)\}\{(x_2y_3) + (a_1b_4)\} + \{(a_1b_3) + (x_4y_2)\}\{(x_1y_3) + (a_4b_2)\}.$$

Taking a_j and x_j to be conjugate complex numbers and also b_j and y_j, for $j = 1, \cdots, 4$, we get an 8 square formula.

23. E. Lucas† stated that the determinant of the 8 equations

$$ax + by + cz + dt + ep + fq + gr + hs = X, \cdots, -hx + \cdots + as = S$$

is the fourth power of $\Delta = a^2 + \cdots + h^2$. To solve the equations, multiply them by $a, \cdots, h$, taken with proper signs, and add. We get

$$\Delta x = aX - bY - cZ - dT - eP - fQ - gR - hS, \cdots,$$
$$\Delta s = hX + \cdots + aS.$$

Squaring these and adding, we get $\Sigma a^2 \cdot \Sigma x^2 = \Sigma X^2$.

24. G. Arnoux‡ argued the impossibility of a 2^n square formula for $n > 3$.

25. Teilhet, de Montessus and Boutin§ gave special numerical examples of $S_rS_r' = S_r''$ for $r = 16$ and $r = 32$, where S_r is a sum of r squares.

26. E. Sadun|| discussed m square formulas of the type (17). Since the product terms in a_pa_t must cancel if $p \neq t$, we have

$$(18) \qquad c_{rp}c_{rt} = -c_{sp}c_{st}, \qquad c_{rp} = \pm c_{st}, \qquad c_{rt} = \mp c_{sp}.$$

Without loss of generality we may assume that the first row and first column of the matrix (c_{ij}) is $c_1, \cdots, c_m$. Then by (18) each diagonal term is $\pm c_1$, whence $m \neq 3$. In the rth and sth rows, $\pm c_{st}$ lies above c_{sp},

* Comptes Rendus, Paris, 104, 1887, 566–7.

† Théorie des nombres, 1891, 294.

‡ Assoc. franç. av. sc., 1896.

§ L'intermédiaire des math., 3, 1896, 259–262.

|| Periodico di Mat., 14, 1899, 125–139; also as a pamphlet of 1896.

and $\mp c_{sp}$ above c_{st}. Hence m must be even. It is assumed that, if $a_i = c_i = 0$ $(i > m/2)$, (17) reduces to a $\frac{1}{2}m$ square formula. Thus if $m = 2^k\omega$, where ω is odd, we would get an ω square formula by continued halving. Hence $\omega = 1$, $m = 2^k$. The impossibility of a 16 square formula is established more simply than in the earlier papers.

27. K. Th. Vahlen* noted the error in Studnička's deduction of the 8 square theorem from the product of two two-rowed determinants and deduced that theorem by use of the product of two three-dimensional determinants [as had Antomari, §22]:

$$\begin{aligned}(aa' + bb' + cc' + dd')(xx' + yy' + zz' + tt')\\ = (ax + by + cz + dt)(a'x' + b'y' + c'z' + d't')\\ + (-b'x + a'y + dz' - ct')(-bx' + ay' + d'z - c't)\\ + (-c'x - dy' + a'z + bt')(-cx' - d'y + az' + b't)\\ + (-d'x + cy' - bz' + a't)(-dx' + c'y - b'z + at').\end{aligned}$$

For $a = a'$, etc., this gives the formula (4) for 4 squares. If a' is the conjugate to a, it gives the 8 square theorem. He gave an analogous, much longer, formula which for $a = a'$, etc., becomes the 8 square formula, but when a' is the conjugate to a, etc., does not yield a 16 square formula.

28. E. Barbette† discussed the 4 and 8 square theorems in connection with magic squares.

* Giornale di Mat., 39, 1901, 181–4.

† Les sommes de p-ièmes puissances . . . , Liège, 1910.

NON-SYMMETRIC KERNELS OF POSITIVE TYPE.

By Dr. Caroline E. Seely.

The properties of continuous symmetric functions of positive type have been studied by Mercer* with especial reference to the uniform convergence of series of their corresponding characteristic functions. In the present paper it is proposed to consider some of the analogous properties of kernels not assumed to be symmetric. Unless otherwise stated, it is assumed that all the kernels $K(s, t)$ considered are real, bounded, and nearly everywhere continuous.

Let us recall some of the well-known theorems for the symmetric case.†

Definition. A function $K(s, t)$ in an interval (a, b) is said to be of positive type if we have

$$(A) \qquad \int_a^b \int_a^b K(s, t)h(s)h(t)dsdt \geq 0$$

for every function $h(s)$ of integrable square.

(I) A necessary and sufficient condition that a symmetric function $K(s, t)$ be of positive type is that its characteristic constants λ_i be all positive.

Corollary 1. If a symmetric kernel is of positive type all its iterated kernels are of positive type.

Corollary 2. All the iterated kernels of even order of any symmetric kernel are of positive type.

(II) The series

$$\sum_{i=0}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i},$$

where the $\varphi_i(s)$ are the characteristic functions and the λ_i the characteristic constants of any symmetric kernel $K(s, t)$, satisfies the condition of mean convergence.‡

(III) The series

$$\sum_{i=0}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i},$$

* Philosophical Transactions, vol. 209 A (1909), p. 415; Goursat, Cours d'Analyse, vol. 3, p. 454.

† Goursat, Cours d'Analyse, vol. 3, p. 449.

‡ Weyl, Mathematische Annalen, vol. 67; Schur, Mathematische Annalen, vol. 66.

where the $\varphi_i(s)$ are the characteristic functions and the λ_i the characteristic constants of a continuous symmetric kernel of positive type, converges uniformly to $K(s, t)$.*

Let the functions $\varphi_i(s)$ and $\psi_i(s)$ be the characteristic functions of a kernel $K(s, t)$ not assumed to be symmetric and let the numbers λ_i be the corresponding characteristic constants. Then we have

$$\int_a^b \varphi_i(s)\psi_i(s)ds = 0 \qquad (\lambda_i \neq \lambda_j).$$

We shall assume the $\varphi_i(s)$ and $\psi_i(s)$ and hence also the λ_i to be all real.

Theorem I. *If $K(s, t)$ is of positive type in an interval (a, b) we have:*

(1) *All the characteristic constants λ_i are positive.*

$$(2) \qquad \begin{aligned} \left(\int_a^b \varphi_i(s)\varphi_j(s)ds\right)^2 &\leq \frac{4\lambda_i\lambda_j}{(\lambda_i + \lambda_j)^2}\int_a^b \varphi_i(s)^2ds \cdot \int_a^b \varphi_j(s)^2ds, \\ \left(\int_a^b \psi_i(s)\psi_j(s)ds\right)^2 &\leq \frac{4\lambda_i\lambda_j}{(\lambda_i + \lambda_j)^2}\int_a^b \psi_i(s)^2ds \cdot \int_a^b \psi_j(s)^2ds. \end{aligned}$$

Let

$$h(s) = \alpha\varphi_i(s) + \beta\varphi_j(s).$$

The condition (A) becomes

$$\begin{aligned} &\int_a^b\int_a^b K(s, t)[\alpha\varphi_i(s) + \beta\varphi_j(s)][\alpha\varphi_i(t) + \beta\varphi_j(t)]dsdt \\ &\quad = \int_a^b \left[\frac{\alpha\varphi_i(s)}{\lambda_i} + \frac{\beta\varphi_j(s)}{\lambda_j}\right][\alpha\varphi_i(s) + \beta\varphi_j(s)]ds \\ &\quad = \frac{\alpha^2}{\lambda_i}\int_a^b \varphi_i(s)^2ds + \alpha\beta\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right)\int_a^b \varphi_i(s)\varphi_j(s)ds + \frac{\beta^2}{\lambda_j}\int_a^b \varphi_j(s)^2ds \geq 0, \end{aligned}$$

and this condition must hold for every choice of α and β. That is, the quadratic form in α and β must be semi-definite, and the conditions for semi-definiteness are

$$\frac{1}{\lambda_i}\int_a^b \varphi_i(s)^2ds > 0, \qquad \frac{1}{\lambda_j}\int_a^b \varphi_j(s)^2ds > 0.$$

$$\left[\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right)\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq 4\frac{1}{\lambda_i}\int_a^b \varphi_i(s)^2ds \cdot \frac{1}{\lambda_j}\int_a^b \varphi_j(s)^2ds.$$

* See Mercer, loc. cit. The assumption of continuity here is essential. In fact, Toeplitz has constructed a bounded symmetric kernel of positive type, continuous except at one point, and such that the series

$$\sum_{i=1}^{\infty} \frac{\varphi_i(s)\varphi_i(t)}{\lambda_i}$$

is not uniformly convergent. See Mathematische Abhandlungen Hermann Amandus Schwarz gewidmet, p. 426.

From these follow

$$\lambda_i > 0, \qquad \lambda_j > 0,$$

$$\left[\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq \frac{4\lambda_i\lambda_j}{(\lambda_i + \lambda_j)^2}\int_a^b \varphi_i(s)^2 ds \int_a^b \varphi_j(s)^2 ds.$$

The second condition of (2) is obtained in exactly similar manner by letting $h(s) = \alpha\psi_i(s) + \beta\psi_j(s)$.

It has just been shown that a necessary condition for a kernel $K(s, t)$ to be of positive type is that the characteristic constants be all positive. It is not, however, true that this condition is also sufficient, as is the case with symmetric kernels. This can be seen from the following example.

Let

$$K(s, t) = \frac{\sin s\ (\sin t + \sin 3t)}{\sqrt{\pi}} + \frac{\sin 2s \sin 2t}{2\sqrt{\pi}}.$$

This kernel has evidently, for the interval $(-\pi, \pi)$, the characteristic constants $\lambda_1 = 1$, $\lambda_2 = 2$ corresponding to the characteristic functions

$$\varphi_1(s) = \frac{\sin s}{\sqrt{\pi}}, \qquad \varphi_2(s) = \frac{\sin 2s}{\sqrt{\pi}},$$

$$\psi_1(s) = \frac{\sin s + \sin 3s}{\sqrt{\pi}}, \qquad \psi_2(s) = \frac{\sin 2s}{\sqrt{\pi}},$$

and no others. Let $h(s) = \sin s + \alpha \sin 3s$, where α is a constant. Then

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K(s, t)h(s)h(t)dsdt = \pi^{3/2}(1 + \alpha),$$

which may be made positive or negative by suitable choice of α.

Theorem II. *If a kernel $K(s, t)$, with more than one characteristic constant, is such that for every positive number n there exists an iterated kernel $K^m(s, t)$, $m > n$, that is of positive type, then if the functions $\varphi_i(s)$, $\psi_i(s)$ are characteristic functions of $K(s, t)$ we have*

$$\int_a^b \varphi_i(s)\varphi_j(s)ds = 0,$$

$$\int_a^b \psi_i(s)\psi_j(s)ds = 0 \qquad (\lambda_i \neq \lambda_j),$$

that is, the $\varphi_i(s)$ and $\psi_i(s)$ each form an orthogonal set.

For the functions $\varphi_i(s)$ and $\psi_i(s)$ are also characteristic functions of the iterated kernel $K^m(s, t)$, corresponding to the characteristic constants λ_i^m. Then from Theorem I (2) we have

$$\left[\int_a^b \varphi_i(s)\varphi_j(s)ds\right]^2 \leq \frac{4\lambda_i^m\lambda_j^m}{(\lambda_i^m + \lambda_j^m)^2}\int_a^b \varphi_i(s)^2ds \int_a^b \varphi_j(s)^2ds,$$

and it is clear that for $\lambda_j > \lambda_i$ the first factor of the right member approaches zero as m increases indefinitely. Whence the constant of the left member is zero, and the functions $\varphi_i(s)$ form an orthogonal set. The same argument holds for the $\psi_i(s)$.

It may be noted that the hypothesis of Theorem II is satisfied by all symmetric kernels, since all their iterated kernels of even order are of positive type (see Corollary 2 of the known theorem (I)).

It might be expected from this theorem that a kernel whose characteristic constants are all positive and whose characteristic functions $\varphi_i(s)$ and $\psi_i(s)$ form two orthogonal sets must be of positive type. That this is not the case is shown by the example given on page 174, for there we have

$$\int_a^b \varphi_1(s)\varphi_2(s)ds = 0, \qquad \int_a^b \psi_1(s)\psi_2(s)ds = 0, \qquad \lambda_1 = 1, \qquad \lambda_2 = 2$$

and yet the kernel is not of positive type.*

Theorem III. *Let $K(s, t)$ be a kernel satisfying the hypothesis of Theorem II, and let the functions $\varphi_{i1}(s), \cdots, \varphi_{im_i}(s)$ and $\psi_{i1}(s), \cdots, \psi_{im_i}(s)$ (where m_i is the number of independent solutions of the homogeneous integral equation corresponding to the characteristic constant λ_i) be a set of its characteristic functions so chosen that*

$$\int_a^b \varphi_{ij}(s)\varphi_{ik}(s)ds = 0, \qquad \int_a^b \psi_{ij}(s)\psi_{ik}(s)ds = 0 \quad (j \neq k).$$

Suppose moreover

$$\int_a^b \varphi_{ik}(s)^2ds < N, \qquad \int_a^b \psi_{ik}(s)^2ds < N,$$

where N is some number independent of i and k. Then the series

$$\sum_{i=1}^{\infty} \frac{\varphi_{i1}(s)\psi_{i1}(t) + \varphi_{i2}(s)\psi_{i2}(t) + \cdots + \varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}$$

satisfies the condition of mean convergence.

This theorem is an immediate consequence of Theorem II, of the definition of mean convergence, and of Schur's theorem.† For

* I am indebted to Dr. Gronwall for this example, as well as for other welcome suggestions.

† The series $\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2}$, where the λ_i are characteristic constants of some kernel $K(s, t)$ each repeated m_i times, is absolutely convergent since $m_i \leq$ the order of λ_i. See Schur, loc. cit., p. 508.

$$\lim_{\mu,\nu=\infty}\int_a^b\int_a^b\left[\sum_{i=1}^{\mu+\nu}\frac{\varphi_{i1}(s)\psi_{i1}(t)+\cdots+\varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}\right.$$

$$\left.-\sum_{i=1}^{\mu}\frac{\varphi_{i1}(s)\psi_{i1}(t)+\cdots+\varphi_{im_i}(s)\psi_{im_i}(t)}{\lambda_i}\right]^2 dsdt$$

$$=\lim_{\mu,\nu=\infty}\sum_{i=\mu}^{\mu+\nu}\int_a^b\int_a^b\frac{\varphi_{i1}(s)^2\psi_{i1}(t)^2+\cdots+\varphi_{im_i}(s)^2\psi_{im_i}(t)^2}{\lambda_i^2}dsdt$$

$$\leq\lim_{\mu,\nu=\infty}\sum_{i=\mu}^{\mu+\nu}\frac{N^2m_i}{\lambda_i^2}=0,$$

that is, the series is convergent in the mean. The final step is from Schur's theorem.

It will be noted that in the proof of Theorem I we did not make full use of the positive character of $K(s, t)$, but only of the fact that condition (A) was satisfied for the restricted class of all functions that are of the form $\alpha\varphi_i(s) + \beta\varphi_j(s)$. It is evident that Theorems II and III are also true under this much weaker assumption.

ELEMENTARY PROPERTIES OF THE STIELTJES INTEGRAL.*

By H. E. Bray.

This paper contains the proofs of certain elementary properties of the Stieltjes integral. The proofs themselves are elementary in character, in the sense that they do not involve point-set theory to any extent. This fact, and the wide possible application of the Stieltjes integral to subjects in mathematical physics, would seem to justify their collection by themselves.

The first theorem deals with the existence of the Stieltjes integral, and is merely introductory. The others lead up to a theorem on the change of the order of integration of an iterated integral, and to a theorem on integration by parts, the latter having been proved and used extensively by Stieltjes.† It is repeated here for the sake of completeness.

Definition 1. Consider two functions, $\varphi(x)$ and $\alpha(x)$, defined at every point of the interval $a \leq x \leq b$. Let (a, b) be divided by the n points $x_1 < x_2 < x_3 < \cdots < x_n$ into $n + 1$ parts such that $x_{i+1} - x_i = \delta_i \leq \delta$, $i = 0, 1, 2, \cdots, n$, $[x_0 = a, x_{n+1} = b]$, and let the points ξ_i be chosen so that $x_i \leq \xi_i \leq x_{i+1}$. The quantity

$$\lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i) \{\alpha(x_{i+1}) - \alpha(x_i)\},$$

if it exists, is called the *Stieltjes integral of* $\varphi(x)$, *with regard to* $\alpha(x)$, *between the limits a and b. It is designated by* $\int_a^b \varphi(x) d\alpha(x)$.

Theorem 1. If $\alpha(x)$ is a function of limited variation, $a \leq x \leq b$, and if $\varphi(x)$ is continuous at every point of (a, b), then $\int_a^b \varphi(x) d\alpha(x)$ exists.

The proof depends on the fact that, since $\alpha(x)$ is of limited variation,

* The writing of the proofs of these theorems was suggested by Professor G. C. Evans; see Cambridge Colloquium Lectures, Lecture V.

† Stieltjes, Recherches sur les fractions continues, Annales de la Faculté des Sciences de Toulouse, 1894.

The concept of the Stieltjes integral has been extended by Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien, 1913. 122, p. 1295.

we can write

$$\alpha(x) = \alpha(a) + P(x) - N(x),$$

where $P(x)$, $N(x)$ are limited non-decreasing functions.*

Evidently

$$\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x_{i+1}) - \alpha(x_i)\} = \sum_{i=0}^{n} \varphi(\xi_i)\{P(x_{i+1}) - P(x_i)\} - \sum_{i=0}^{n} \varphi(\xi_i)\{N(x_{i+1}) - N(x_i)\}.$$

It is sufficient, therefore, to prove the existence of

$$\int_a^b \varphi(x)d\beta(x) = \lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i)\{\beta(x_{i+1}) - \beta(x_i)\},$$

where $\beta(x)$ is limited and non-decreasing in (a, b).

Let $M_i(m_i)$ be the maximum (minimum) value of $\varphi(x)$ in the subinterval $x_i \leq x \leq x_{i+1}$, and let

$$S_n = \sum_{i=0}^{n} M_i\{\beta(x_{i+1}) - \beta(x_i)\}, \qquad s_n = \sum_{i=0}^{n} m_i\{\beta(x_{i+1}) - \beta(x_i)\}.$$

Evidently

$$S_n \geq s_n \tag{1}$$

if both sums are formed according to the same mode of division of (a, b). Also

$$M\{\beta(b) - \beta(a)\} \geq S_n, \qquad m\{\beta(b) - \beta(a)\} \leq s_n, \tag{2}$$

where $M(m)$ is the maximum (minimum) value of φ in (a, b), for

$$\sum_{i=0}^{n} M\{\beta(x_{i+1}) - \beta(x_i)\} \geq \sum_{i=0}^{n} M_i\{\beta(x_{i+1}) - \beta(x_i)\}.$$

By applying inequalities (2) to subintervals of (a, b) instead of to (a, b) itself, it is easily seen that, if $S_{n_1+n_2}$ is a sum formed by adding n_2 points of division to those of S_{n_1},

$$S_{n_1+n_2} \leq S_{n_1}, \qquad s_{n_1+n_2} \geq s_n\,. \tag{3}$$

* de la Vallée Poussin, Cours d'analyse, Vol. I, third edition, p. 72 et seq. The facts are as follows: Regard the sum $\tau = \sum_{i=0}^{n} |\alpha(x_{i+1}) - \alpha(x_i)|$, taken over the closed interval (a, x), as made up of terms of two classes, viz.: those corresponding (i) to positive (ii) to negative differences. Let $P(x)$, $N(x)$ be the upper limits, respectively, of the sums of the two classes of terms, for all modes of division of (a, x), and let $T(x)$ be, similarly, the limit of τ. If $T(x)$ exists, $\alpha(x)$ is, by definition, of limited variation in (a, x). The existence of $T(x)$, called the total variation function of $\alpha(x)$, implies that of both $P(x)$ and $N(x)$. $T(x) = P(x) + N(x)$. These three are, of course, non-decreasing functions, and $\alpha(x) = \alpha(a) + P(x) - N(x)$.

We can now show that every S, no matter how formed, is at least as great as every s. For suppose $S_{n_1} < s_{n_2}$; then, by (3),

$$S_{n_1+n_2} \leq S_{n_1} < s_{n_2} \leq s_{n_1+n_2},$$

which is impossible, in view of (1).

The numbers S have a lower limit L, and the numbers s an upper limit l. To show that

$$L = l = \int_a^b \varphi(x)d\beta(x)$$

we have only to note that

$$S_n \geq \sum_{i=0}^{n} \varphi(\xi_i)\{\beta(x_{i+1}) - \beta(x_i)\} \geq s_n$$

and to show that, by taking δ small enough, $S_n - s_n$ can be made as small as we please.

Since $\varphi(x)$ is continuous in a closed interval, and therefore uniformly continuous, it is possible to choose δ so small that, for any ϵ assigned in advance,

$$M_i - m_i \leq \frac{\epsilon}{\beta(b) - \beta(a)} \qquad i = 0, 1, 2, \cdots, n.$$

Hence

$$S_n - s_n = \sum_{i=0}^{n} \{M_i - m_i\}\{\beta(x_{i+1}) - \beta(x_i)\} \leq \epsilon.$$

The theorem is thus proved.

In some of the theorems which follow it is convenient to be able to reduce the proof to the discussion of finite sums instead of integrals. We therefore introduce the following lemma:

Theorem 2.* If $\alpha(s)$ is of limited variation, $a \leq s \leq b$, and if

$$\int_a^b \varphi d\alpha = \lim_{\delta=0} \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\}$$

exists, then

$$\left| \int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\} \right| \leq O_\delta T(b)$$

where $T(b)$ is the total variation of α in the interval (a, b), O_δ is the greatest oscillation of $\varphi(s)$ in the $n + 1$ intervals (s_i, s_{i+1}), and $s_i \leq \xi_i \leq s_{i+1}$.

Consider the sum

$$\sum_{i=0}^{n'} \varphi(\xi_i')\{\alpha(s_{i+1}') - \alpha(s_i')\},$$

formed by a set of points (s_i') which includes the set (s_i).

* This proof of Theorem 2 was suggested by Mr. H. L. Smith and is more ingenious and somewhat shorter than that of the writer, for which it is therefore substituted.

Evidently

$$\left|\int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\}\right|$$
$$\leq \left|\int_a^b \varphi(d\alpha) - \sum_{i=0}^{n'} \varphi(\xi_i')\{\alpha(s_{i+1}') - \alpha(s_i')\}\right|$$
$$+ \left|\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\} - \sum_{i=0}^{n'} \varphi(\xi_i')\{\alpha(s_{i+1}') - \alpha(s_i')\}\right|.$$

But by choosing a small enough norm for the points s_i' the first term of the right-hand member, by our hypothesis, can be made as small as we please, and, since the second term is at most equal to $O_\delta \int |d\alpha| = O_\delta T(b)$,

$$\left|\int_a^b \varphi d\alpha - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(s_{i+1}) - \alpha(s_i)\}\right| \leq O_\delta T(b).$$

Definition 2. Consider a function $\alpha(x, s)$ which, for every value of x, $c \leq x \leq d$, is of limited variation in s, $a \leq s \leq b$. Let $T(x, s)$ be the corresponding total variation function in s, for a given value of x. Evidently $T(x, s) \leq T(x, b)$.

If $T(x, b)$ is a limited function of x, i. e., if a positive constant K can be found, such that

$$T(x, b) < K \qquad c \leq x \leq d$$

then $\alpha(x, s)$ is said to be a *function of uniformly limited variation in s for all values of x in* (c, d).

Theorem 3. If $\varphi(s)$ is continuous, $a \leq s \leq b$, and if $\alpha(x, s)$ is a function of uniformly limited variation in s for all values of x in the interval $c \leq x \leq d$, and if at every point x in (c, d), a set of values of s,* dense in (a, b) and including a and b, can be found, such that for each value of s in the set $\alpha(x, s)$ is continuous in x, then $\Phi(x) = \int_a^b \varphi(s) d_s\alpha(x, s)$ is continuous, $c \leq x \leq d$.

By Theorem 2, a positive constant K exists such that

$$\left|\Phi(x) - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x, s_{i+1}) - \alpha(x, s_i)\}\right| \leq O_\delta K$$

and

$$\left|\Phi(x + \Delta x) - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x + \Delta x, s_{i+1}) - \alpha(x + \Delta x, s_i)\}\right| \leq O_\delta K.$$

* This set of values of s need not of course be independent of x.

Therefore

$$\left| [\Phi(x + \Delta x) - \Phi(x)] - \left[\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x + \Delta x, s_{i+1}) - \alpha(x + \Delta x, s_i)\} - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x, s_{i+1}) - \alpha(x, s_i)\} \right] \right| \leq 2O_\delta K.$$

Now, if ϵ be assigned, a norm δ can be found, so small that, since $\varphi(s)$ is uniformly continuous, $O_\delta < \frac{1}{4} \cdot \frac{\epsilon}{K}$ and therefore

$$|\Phi(x + \Delta x) - \Phi(x)| \leq \tfrac{1}{2}\epsilon + \left| \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x + \Delta x, s_{i+1}) - \alpha(x + \Delta x, s_i)\} - \sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x, s_{i+1}) - \alpha(x, s_i)\} \right|.$$

This inequality holds for all modes of division of the interval $a \leq s \leq b$ of norm δ. Let us select a mode, of norm δ, such that the points of division, $a, s_1, s_2, \cdots, s_n, b$ are all points at which $\alpha(x, s)$ is continuous in x. Then the term between absolute value signs is merely the difference of the continuous function of x, $\sum_{i=0}^{n} \varphi(\xi_i)\{\alpha(x, s_{i+1}) - \alpha(x, s_i)\}$, taken at the values x and $x + \Delta x$, and therefore can be made as small as we please ($< \epsilon/2$) by taking Δx small enough; whence it follows that

$$|\Phi(x + \Delta x) - \Phi(x)| < \epsilon$$

and the theorem is proved.

Theorem 4. If $\gamma(x)$ is a function of limited variation, $c \leq x \leq d$, and if $\alpha(x, s)$ is of uniformly limited variation in s, $a \leq s \leq b$, for every value of x, $c \leq x \leq d$, and continuous in x for every value of s, then

$$\Psi(s) = \int_c^d \alpha(x, s) d\gamma(x)$$

is a function of limited variation, $a \leq s \leq b$.

It is required to show that a positive constant L can be found, such that, for any mode of division whatever of the interval (a, b), by points $s_0 = a, s_1, s_2, \cdots, s_m, s_{m+1} = b$,

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq L.$$

By expressing $\gamma(x)$ as the difference of two limited, non-decreasing functions, $\gamma_1(x)$ and $\gamma_2(x)$, we have:

$$\Psi(s) = \int_c^d \alpha(x, s) d\gamma_1(x) - \int_c^d \alpha(x, s) d\gamma_2(x),$$

and since the difference of two functions of limited variation is also of limited variation, our theorem will be proved if we can show that

$$\Psi(s) = \int_c^d \alpha(x, s) d\bar{\gamma}(x)$$

is of limited variation in the special case in which $\bar{\gamma}(x)$ is a limited, non-decreasing function. It will be shown that

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq 2K[\bar{\gamma}(d) - \bar{\gamma}(c)],$$

where K, as before, is the upper bound of the total variation of $\alpha(x, s)$.

Regarding the quantities $s_1, s_2, \cdots, s_m$, as fixed, and applying the approximation formula of Theorem 2, we obtain

$$\left| \Psi(s_j) - \sum_{i=0}^{n} \alpha(\xi_i, s_j)\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} \right| \leq O_\delta^{[j]}[\bar{\gamma}(d) - \bar{\gamma}(c)]$$
$$j = 0, 1, 2, \cdots (m + 1),$$

where $O_\delta^{[j]}$ is the greatest oscillation of $\alpha(x, s_j)$ in any of the intervals $x_i \leq x \leq x_{i+1}$, $i = 0, 1, 2, \cdots, n$.

Since $\alpha(x, s)$ is continuous in x for each of the values $s = a, s_1, s_2, \cdots, s_m, b$, a single mode of division of the interval $c \leq x \leq d$ can be chosen, of norm δ so small that

$$O_\delta^{[j]} < \frac{K}{2(m + 1)},$$

this inequality holding, at once, for all of the given values of s_j. Hence

$$\begin{aligned} |\Psi(s_{j+1}) - \Psi(s_j)| \leq & \left| \sum_{i=0}^{n} \{\alpha(\xi_i, s_{j+1}) - \alpha(\xi_i, s_j\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} \right| \\ & + \frac{K}{m + 1}[\bar{\gamma}(d) - \bar{\gamma}(c)] \qquad j = 0, 1, 2, \cdots, m. \end{aligned} \tag{1}$$

Now

$$\begin{aligned} & \left| \sum_{i=0}^{n} \{\alpha(\xi_i, s_{j+1}) - \alpha(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} \right| \\ & = \left| \sum_{i=0}^{n} \{P(\xi_i, s_{j+1}) - N(\xi_i, s_{j+1}) - P(\xi_i, s_j) + N(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} \right| \\ & \leq \left| \sum_{i=0}^{n} \{P(\xi_i, s_{j+1}) + N(\xi_i, s_{j+1}) - P(\xi_i, s_j) - N(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\} \right|, \end{aligned}$$

since

$$N(\xi_i, s_{j+1}) \geq N(\xi_i, s_j),$$

and this last expression can be written:

$$\sum_{i=0}^{n} \{T(\xi_i, s_{j+1}) - T(\xi_i, s_j)\}\{\bar{\gamma}(x_{i+1}) - \bar{\gamma}(x_i)\}.$$

Substituting this result in (1) and adding together the results for $j = 0, 1, 2, \cdots, m$, we obtain

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq \sum_{j=0}^{m}\sum_{i=0}^{n} \{T(\xi_i, s_{j+1}) - T(\xi_i, s_j)\}\{\overline{\gamma}(x_{i+1}) - \overline{\gamma}(x_i)\} + K[\overline{\gamma}(d) - \overline{\gamma}(c)].$$

Changing the order of summation in the double sum, we obtain

$$\begin{aligned}\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| &\leq \sum_{i=0}^{n} \{T(\xi_i, b) - T(\xi_i, a)\}\{\overline{\gamma}(x_{i+1}) - \overline{\gamma}(x_i)\} \\ &\qquad + K[\overline{\gamma}(d) - \overline{\gamma}(c)] \\ &\leq K\sum_{i=0}^{n} \{\overline{\gamma}(x_{i+1}) - \overline{\gamma}(x_i)\} + K\{\overline{\gamma}(d) - \overline{\gamma}(c)\}\end{aligned}$$

since α is of uniformly limited variation in s. Hence finally

$$\sum_{j=0}^{m} |\Psi(s_{j+1}) - \Psi(s_j)| \leq 2K[\overline{\gamma}(d) - \overline{\gamma}(c)].$$

Theorem 5. If $\varphi(s)$ is continuous, $a \leq s \leq b$; if $\overline{\gamma}(x)$ is of limited variation, $c \leq x \leq d$; and if $\alpha(x, s)$ is continuous in x for all values of s in (a, b) and of uniformly limited variation in s for all values of x in (c, d), then the integrals

$$\int_c^d \left[\int_a^b \varphi(s) d_s\alpha(x, s)\right] d\gamma(x), \qquad \int_a^b \varphi(s) d_s \int_c^d \alpha(x, s) d\gamma(x)$$

exist and are equal.

By Theorem 3 $\Phi(x) = \int_a^b \varphi(s) d_s\alpha(x, s)$ is continuous, and, by Theorem 4, $\Psi(s) = \int_c^d \alpha(x, s) d\gamma(x)$ is of limited variation. Hence, by Theorem 1, $\int_c^d \Phi(x) d\gamma(x)$ and $\int_a^d \varphi(s) d\Psi(s)$ exist.

It is to be shown that, no matter how small a positive constant ϵ be assigned,

$$\left| \int_c^d \Phi(x) d\gamma(x) - \int_a^b \varphi(s) d\Psi(s) \right| < \epsilon.$$

The following relation, which is easily verified, will be used:

$$\begin{aligned}&\sum_{j=0}^{m}\left[\sum_{i=0}^{n} \varphi(s_i)\{\alpha(x_j, s_{i+1}) - \alpha(x_j, s_i)\}\right][\gamma(x_{j+1}) - \gamma(x_j)] \\ (1) \qquad &= \sum_{i=0}^{n} \varphi(s_i)\left[\sum_{j=0}^{m} \alpha(x_j, s_{i+1})\{\gamma(x_{j+1}) - \gamma(x_j)\} \right. \\ &\qquad \left. - \sum_{j=0}^{m} \alpha(x_j, s_i)\{\gamma(x_{j+1}) - \gamma(x_j)\}\right].\end{aligned}$$

Since $\Phi(x)$ is continuous, $c \leq x \leq d$, it is possible to divide (c, d) into $m + 1$ parts, of norm δ_x so small that, by Theorem 2,

$$\left| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m} \Phi(x_j) \{ \gamma(x_{j+1}) - \gamma(x_j) \} \right| < \frac{\epsilon}{4}. \tag{2}$$

It is also possible to divide (a, b) into $n + 1$ parts, of norm δ_s so small that

$$\begin{aligned} \Bigg| \sum_{j=0}^{m} \Phi(x_j) \{ \gamma(x_{j+1}) - \gamma(x_j) \} - \sum_{j=0}^{m} \Bigg[\sum_{i=0}^{n} \varphi(s_i) \{ \alpha(x_j, s_{i+1}) \\ - \alpha(x_j, s_i) \} \Bigg] [\gamma(x_{j+1}) - \gamma(x_j)] \Bigg| < \frac{\epsilon}{4}. \end{aligned} \tag{3}$$

In fact, since $\varphi(s)$ is uniformly continuous in (a, b) and $\alpha(x, s)$ is of uniformly limited variation in s for every x, by Theorem 2, δ_s can be chosen so that, for every x,

$$\left| \Phi(x) - \sum_{i=0}^{n} \varphi(s_i) \{ \alpha(x, s_{i+1}) - \alpha(x, s_i) \} \right| < \frac{\epsilon/4}{\tau(d)},$$

where $\tau(d)$ is the total variation of $\gamma(x)$ in (c, d).

From (2) and (3) it follows that, for norms δ_x and δ_s,

$$\begin{aligned} \Bigg| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m} \Bigg[\sum_{i=0}^{n} \varphi(s_i) \{ \alpha(x_j, s_{i+1}) \\ - \alpha(x_j, s_i) \} \Bigg] [\gamma(x_{j+1}) - \gamma(x_j)] \Bigg| < \frac{\epsilon}{2}, \end{aligned} \tag{4}$$

and by virtue of the uniformity referred to above, (4) is true, a fortiori, for any smaller norms.

Now let (a, b) be divided, norm $\delta_s'(< \delta_s)$, into $n' + 1$ parts so that

$$\begin{aligned} \Bigg| \int_a^b \varphi(s) d_s \int_c^d \alpha(x, s) d\gamma(x) - \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) \\ - \int_c^d \alpha(x, s_i) d\gamma(x) \Bigg] \Bigg| < \frac{\epsilon}{4} \end{aligned} \tag{5}$$

and let (c, d) be divided, norm $\delta_x'(< \delta_x)$, into $m' + 1$ parts so that

$$\begin{aligned} \Bigg| \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) - \int_c^d \alpha(x, s_i) d\gamma(x) \Bigg] \\ - \sum_{i=0}^{n'} \varphi(s_i) \Bigg[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{ \gamma(x_{j+1}) - \gamma(x_j) \} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{ \gamma(x_{j+1}) \\ - \gamma(x_j) \} \Bigg] \Bigg| < \frac{\epsilon}{4}. \end{aligned} \tag{6}$$

This can be done, for δ_x' can be chosen so small that

$$\left| \int_c^d \alpha(x, s_i) d\gamma(x) - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{\gamma(x_{j+1}) - \gamma(x_j)\} \right| < \frac{\epsilon}{8M(n'+1)}$$

$$i = 0, 1, 2, \cdots, n',$$

where $M > |\varphi(s)|$. Hence

$$\left| \left[\int_c^d \alpha(x, s_{i+1}) d\gamma(x) - \int_c^d \alpha(x, s_i) d\gamma(x) \right] - \left[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{\gamma(x_{j+1}) - \gamma(x_j)\} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{\gamma(x_{j+1}) - \gamma(x_j)\} \right] \right| < \frac{\epsilon}{4M(n'+1)}.$$

From (5) and (6) it follows that, for norms δ_x' and δ_s',

$$\left| \int_a^b \varphi(s) d\Psi(s) - \sum_{i=0}^{n'} \varphi(s_i) \left[\sum_{j=0}^{m'} \alpha(x_j, s_{i+1}) \{\gamma(x_{j+1}) - \gamma(x_j)\} - \sum_{j=0}^{m'} \alpha(x_j, s_i) \{\gamma(x_{j+1}) - \gamma(x_j)\} \right] \right| < \frac{\epsilon}{2}.$$

From (4) it follows, since $\delta_x' < \delta_x$ and $\delta_s' < \delta_s$, that

$$\left| \int_c^d \Phi(x) d\gamma(x) - \sum_{j=0}^{m'} \left[\sum_{i=0}^{n'} \varphi(s_i) \{\alpha(x_j, s_{i+1}) - \alpha(x_j, s_i)\} \right] \times [\gamma(x_{j+1}) - \gamma(x_j)] \right| < \frac{\epsilon}{2}.$$

and, using equation (1),

$$\left| \int_c^d \Phi(x) d\gamma(x) - \int_a^b \varphi(s) d\Psi(s) \right| < \epsilon.$$

Theorem 6. If $\varphi(x)$ is continuous, $a \leq x \leq b$, and if $\alpha(x)$ is of limited variation in the same interval, then the integral $\int_a^b \alpha(x) d\varphi(x)$ exists and is equal to

$$\alpha(x)\varphi(x) \Big|_a^b - \int^b \varphi(x) d\alpha(x).$$

Consider the sum:

$$\sum_{i=0}^{n} \alpha(\xi_i) \{\varphi(x_{i+1}) - \varphi(x_i)\}$$
$$= \alpha(\xi_0)\{\varphi(x_1) - \varphi(a)\} + \alpha(\xi_1)\{\varphi(x_2) - \varphi(x_1)\} + \cdots$$
$$+ \alpha(\xi_{n-1})\{\varphi(x_n) - \varphi(x_{n-1})\} + \alpha(\xi_n)\{\varphi(b) - \varphi(x_n)\},$$

where $x_i \leq \xi_i \leq x_{i+1}$, $x_{i+1} - x_i \leq \delta$, $x_0 = a$, $x_{n+1} = b$ and also, of course, $\xi_i \leq x_{i+1} \leq \xi_{i+1}$.

By adding and subtracting $\alpha(b)\varphi(b) - \alpha(a)\varphi(a)$ and regrouping the terms we obtain

$$\sum_{i=0}^{n} \alpha(\xi_i)\{\varphi(x_{i+1}) - \varphi(x_i)\} = -\varphi(a)\{(\xi_0) - \alpha(a)\} - \varphi(x_1)\{\alpha(\xi_1) - \alpha(\xi_0)\}$$
$$- \cdots - \varphi(x_n)\{\alpha(\xi_n) - \alpha(\xi_{n-1})\} - \varphi(b)\{\alpha(b) - \alpha(\xi_n)\}$$
$$+ \alpha(b)\varphi(b) - \alpha(a)\varphi(a).$$
$$= \alpha(x)\varphi(x)\Big|_a^b - \sum_{i=0}^{n+1} \varphi(x_i)\{\alpha(\xi_i) - \alpha(\xi_{i-1})\}.$$

In this last summation ξ_{-1} denotes the point a, and ξ_{n+1} the point b.

But

$$\lim_{\delta=0} \sum_{i=0}^{n+1} \varphi(x_i)\{\alpha(\xi_i) - \alpha(\xi_{i-1})\} = \int_a^b \varphi(x)d\alpha(x)$$

for $\xi_i - \xi_{i-1} \leq 2\delta$. Hence

$$\lim_{\delta=0} \sum_{i=0}^{n} \alpha(\xi_i)\{\varphi(x_{i+1}) - \varphi(x_i)\} = \int_a^b \alpha(x)d\varphi(x)$$

exists and is equal to

$$\alpha(x)\varphi(x)\Big|_a^b - \int_a^b \varphi(x)d\alpha(x).$$

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A KINEMATICAL PROPERTY OF RULED SURFACES.*

By J. K. Whittemore.

Let S be any ruled surface, not developable, with real rulings, C a real curve of S, not a straight line, intersecting all rulings considered, and g any such ruling; let the coördinates of a point of C be x_0, y_0, z_0; suppose C rectifiable and let its arc be v; suppose C has at every point a definite principal trihedral, and let α, β, γ be the direction cosines of the tangent, l, m, n those of the principal normal, λ, μ, ν those of the binormal; let R and T be the radii of curvature and torsion respectively; let ψ be the angle made by the direction chosen as positive on g with the osculating plane of C measured towards the positive binormal, φ the angle of the plane of g and the binormal measured from the rectifying plane towards the normal plane. (If g coincides with the binormal φ is not determined.) The surface S is given by

$$(1)\qquad x = x_0 + uL, \qquad L = \alpha\cos\psi\cos\varphi + l\cos\psi\sin\varphi + \lambda\sin\psi,$$

with similar equations for y and z, where u is the length measured on g from C in the positive direction. Supposing ψ and φ to have finite first derivatives with respect to v the linear element of S is given by

$$(2)\qquad \begin{aligned} ds^2 = (du + \cos\psi\cos\varphi dv)^2 + \{&\cos^2\psi\sin^2\varphi + \sin^2\psi - 2u[\sin\psi\cos\varphi\psi' \\ &+ \cos\psi\sin\varphi(\varphi' + 1/R)] + u^2[((\sin\varphi)/T - \psi')^2 \\ &+ (\cos\psi\{\varphi' + 1/R\} + (\sin\psi\cos\varphi)/R)^2]\}dv^2. \end{aligned}$$

The coefficient of u^2 in (2) does not vanish identically since S is not developable.† The linear element of a ruled surface may be given the form‡

$$(3)\qquad ds^2 = du_1^2 + [(u_1 - a)^2 + b^2]dv_1^2,$$

where the curves (v_1), that is v_1 constant, are the rulings, u_1 is the length measured on a ruling from the orthogonal trajectory, $u_1 = 0$; the function a, depending on v_1 alone, is the distance from this trajectory to the central point of the ruling (v_1), so that the equation of the line of striction σ is $u_1 = a$, and b also depending on v_1 alone is the parameter of distribution.

* Presented to the American Mathematical Society, April 28, 1917.

† Darboux, Théorie des surfaces, vol. 3, p. 294.

‡ Darboux, vol. 1, 2d ed., p. 122.

Comparing (2) and (3) it is evident that the necessary and sufficient condition that C be σ is

$$(4) \qquad \sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R) = 0.$$

The parameter of distribution of S is*

$$(5) \qquad b = -\mid x_0'MN' \mid /\Sigma L'^2,$$

where L, M, N are the coefficients of u in (1), and the numerator of the second member is a determinant of which we have written the principal diagonal. If C is σ and is not a geodesic of S, so that $\varphi \neq 0$, (5) gives, using (4),

$$(6) \qquad \frac{1}{b} = \frac{1}{T} - \frac{\psi'}{\sin\varphi}.$$

Supposing now that C is the line of striction σ of a ruled surface S, and that C is not a geodesic of S, let ξ, η, ζ be the coördinates of any point of the ruling g referred to the principal trihedral of C, and referred also to axes fixed in space and coincident with the principal trihedral; let d and δ be differentials referring to the moving and fixed axes respectively. We have†

$$\frac{\delta\xi}{dv} = \frac{d\xi}{dv} + 1 - \frac{\eta}{R}, \qquad \frac{\delta\eta}{dv} = \frac{d\eta}{dv} + \frac{\xi}{R} + \frac{\zeta}{T}, \qquad \frac{\delta\zeta}{dv} = \frac{d\zeta}{dv} - \frac{\eta}{T},$$

$$\xi = u\cos\psi\cos\varphi, \qquad \eta = u\cos\psi\sin\varphi, \qquad \zeta = u\sin\psi.$$

From these equations

$$\frac{\delta\xi}{dv} = 1 - u[\sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R)].$$

$$\frac{\delta\eta}{dv} = u[\sin\psi(1/T - \sin\varphi\psi') + \cos\psi\cos\varphi(\varphi' + 1/R)].$$

$$\frac{\delta\zeta}{dv} = -u\cos\psi((\sin\varphi)/T - \psi').$$

We find from the preceding equations

$$\eta\frac{\delta\eta}{dv} + \zeta\frac{\delta\zeta}{dv} = u^2\cos\psi\cos\varphi[\sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R)],$$

$$\zeta\frac{\delta\eta}{dv} - \eta\frac{\delta\zeta}{dv} = u^2\left[\frac{\cos^2\psi\sin^2\varphi + \sin^2\psi}{T} - \sin\varphi\psi'\right.$$

* Darboux, vol. 3, p. 302.

† Eisenhart, Differential Geometry, p. 32.

$$+ \sin\psi\cos\psi\cos\varphi\left(\varphi' + \frac{1}{R}\right)\Big] = u^2\Big[(\cos^2\psi\sin^2\varphi$$
$$+ \sin^2\psi)\left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right) + \frac{\sin\psi\cos\psi}{\sin\varphi}\Big(\sin\psi\cos\varphi\psi'$$
$$+ \cos\psi\sin\varphi\left\{\varphi' + \frac{1}{R}\right\}\Big)\Big].$$

Using (4) and (6) these give*

$$(7) \qquad \frac{\delta\xi}{dv} = 1, \qquad \eta\frac{\delta\eta}{dv} + \zeta\frac{\delta\zeta}{dv} = 0, \qquad \zeta\frac{\delta\eta}{dv} - \eta\frac{\delta\zeta}{dv} = \frac{\eta^2 + \zeta^2}{b}.$$

Equations (7) hold in the case, excluded from the preceding discussion, where σ is a curved geodesic of S, and also with the proper modifications in the definitions of the coördinates where σ is a straight line.

From (7) follows

THEOREM I. *Every ruled surface S may be generated by a radius fixed in a sphere whose center moves with unit velocity along the line of striction and which turns about the tangent to this curve from the binormal towards the principal normal with angular velocity equal to the reciprocal of the parameter of distribution.*

There follow two corollaries:

If two ruled surfaces have the same line of striction and equal parameters of distribution their rulings intersect on the line of striction at a constant angle.
If a sphere moves so that it always rotates about the tangent to the locus of its center every radius fixed in the sphere generates a ruled surface whose line of striction is the locas of the center; all such surfaces have the same parameter of distribution.

From (6), if σ is not a geodesic of S, follows

THEOREM II. *A necessary and sufficient condition that the parameter of distribution of a ruled surface S be equal to the radius of torsion of the line of striction at the corresponding point is that the ruling form a constant angle with the binormal of the line of striction at the point of intersection.*

The case where the line of striction is a geodesic is included in Theorem II, for it may be shown that in this case the parameter of distribution is equal to the radius of torsion of the line of striction when and only when the ruling coincides with the binormal.

From (4) it follows if $\psi' = 0$, $\sin\varphi \neq 0$ and S is not a binormal surface (*i. e.*, $\cos\psi \neq 0$), that for all ruled surfaces S, such that $b = T$ on σ, except binormal surfaces,

$$(8) \qquad \psi' = 0, \qquad \varphi' + 1/R = 0.$$

Such a surface we call a T surface. From Theorem I it follows that the

* Equations (7) follow from equations given by Cesàro, Geometria intrinseca, pp. 134, 135. Cesàro makes no application similar to that here given.

osculating plane of σ is fixed in the generating sphere of S, and conversely, that if the osculating plane of σ is fixed in the sphere S is a T surface or a binormal surface. A T surface is uniquely determined by choosing any curve C as σ, and any line intersecting C, not a binormal, as a ruling.

We consider deformations connected with T surfaces, proving that any ruled surface S, on which σ is not a geodesic, is applicable with correspondence of rulings to a T surface, and that a T surface, if $\psi \neq 0$, may be continuously deformed while remaining a T surface. For any surface S, on which σ is not a geodesic, we have from (2) and (4)

$$ds^2 = (du + \cos\psi \cos\varphi dv)^2 + (\cos^2\psi \sin^2\varphi + \sin^2\psi)\left[1 + u^2\left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right)^2\right]dv^2.$$

Using subscripts for a T surface S_1,

$$ds_1^2 = (du + \cos\psi_1 \cos\varphi_1 dv)^2 + (\cos^2\psi_1 \sin^2\varphi_1 + \sin^2\psi_1)\left(1 + \frac{u^2}{T_1^2}\right)dv^2.$$

For applicability of S and S_1 the lines of striction must correspond. Necessary and sufficient conditions for applicability are

$$(9) \qquad \cos\psi_1 \cos\varphi_1 = \cos\psi \cos\varphi, \qquad \frac{1}{T_1^2} = \left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right)^2,$$

with $\psi_1' = \varphi_1' + 1/R_1 = 0$, and (4) holding for S. By differentiating the first of (9), we find on reduction by means of (4)

$$\frac{\cos\psi_1 \sin\varphi_1}{R_1} = \frac{\cos\psi \sin\varphi}{R}.$$

If S is given, S_1 may be determined to satisfy these conditions as follows: ψ_1 may be chosen arbitrarily subject to the condition, $|\cos\varphi_1| = |\cos\psi \cos\varphi \sec\psi_1| \leqq 1$. If we consider the deformation of a given part of S it is clear that the numerical value of ψ_1 is at most equal to the smallest numerical value of the angle of the ruling and the tangent to σ on that part of S considered; if this value is zero the only possible choice is $\psi_1 = 0$, and σ_1 is an asymptotic line of S_1. When ψ_1 is chosen, $\cos\varphi_1$ is determined, then the sign of $\sin\varphi_1$, hence φ_1 and the positive R_1. The torsion of σ_1 is determined except as to sign, but if the deformation of S into S_1 is continuous this sign is also fixed. Then σ_1 and φ_1 and consequently S_1 are determined by the choice of ψ_1 except for the sign of T_1 in non-continuous deformation. The statements made concerning the continuous deformation of a T surface while remaining a T surface follow from the preceding discussion. It is obvious that the lines of striction of all T surfaces applicable to a given surface S have the same torsion numerically at corresponding points. If the parameter of distribution of S is constant the line of striction of an applicable T surface is a curve of constant torsion.

YALE UNIVERSITY, 1918.

SYSTEMS OF LINEAR INEQUALITIES.

By Lloyd L. Dines.

The existence of a solution of the system of inequalities

$$
(1) \qquad
\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n > 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n > 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n > 0,
\end{array}
$$

and the character of the solution in case one exists, obviously depend upon the matrix of the coefficients. There seems however to be no terminology available for the characterization of the solution in terms of this matrix, and no well-recognized algorithm for the actual determination of the solution. This is in marked contrast to the analogous situation in the case of the system of equations formed by replacing the symbol $>$ by the symbol $=$ in (1). In the present paper a study is made of the system (1) and also of a system of non-homogeneous inequalities from the point of view of the matrix of coefficients. The central feature is a concept analogous to the rank of a matrix, which we call the *inequality-rank* or *I-rank* of the matrix. In terms of this concept the principal results are expressed in Theorems I–IV.

1. The I-rank of a Matrix. Let

$$
M = \left\| \begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdot & \cdot & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array} \right\|
$$

be a matrix whose mn elements are real. For our purpose it is important to distinguish the case in which at least one column of M has elements all of the same sign. We make the

Definition. *A matrix will be said to be I-positive (or I-negative) with respect to a given one of its columns if all elements of that column are positive (or negative). In either case the matrix will be said to be I-definite with respect to that column. A matrix will be said to be I-positive (or I-negative, or I-definite) if it possesses a column with respect to which it is I-positive (or I-negative, or I-definite).*

Suppose the matrix M is not I-definite with respect to the rth column. Then the elements of that column can be divided into three classes, viz.,

those which are positive: a_{ir}, $i = i_1, i_2, \cdots, i_P$;

those which are negative: a_{jr}, $j = j_1, j_2, \cdots, j_N$;

those which are zero: a_{kr}, $k = k_1, k_2, \cdots, k_Z$;

the number of elements in the respective classes being represented by P, N, and Z.

Consider a matrix $M_1^{(r)}$ derived from M as follows:

To each pair of elements a_{ir}, a_{jr}, the first positive and the second negative, corresponds one row of the derived matrix, the elements of which are second order determinants

$$\begin{vmatrix} a_{ir} & a_{i1} \\ a_{jr} & a_{j1} \end{vmatrix}, \quad \begin{vmatrix} a_{ir} & a_{i2} \\ a_{jr} & a_{j2} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} a_{ir} & a_{ir-1} \\ a_{jr} & a_{jr-1} \end{vmatrix}, \quad \begin{vmatrix} a_{ir} & a_{ir+1} \\ a_{jr} & a_{jr+1} \end{vmatrix}, \quad \cdots, \quad \begin{vmatrix} a_{ir} & a_{in} \\ a_{jr} & a_{jn} \end{vmatrix}.$$

To each zero element a_{kr} corresponds one row of the derived matrix

$$a_{k1}, \quad a_{k2}, \quad \cdots, \quad a_{kr-1}, \quad a_{kr+1}, \quad \cdots, \quad a_{kn}.$$

The matrix $M_1^{(r)}$ will then consist of the rows so formed, their number being $P \cdot N + Z$. The sequential order of the rows will be determined by the convention that: (1) each (ij) row precedes every (k) row; and (2) of two (ij) rows that one precedes which has the smaller i or (in case of equal i's) that one which has the smaller j.

The matrix $M_1^{(r)}$ is then well-defined provided the matrix M is not I-definite with respect to the rth column. For the sake of uniformity it will be convenient to define $M_1^{(r)}$ in case M is I-definite with respect to the rth column, as the matrix of one row and $n - 1$ columns, all elements of which are $+1$ or -1 according as M is I-positive or I-negative with respect to the rth column. The matrix $M_1^{(r)}$ will be called the I-*complement* of the rth column of M, and may be represented by

$$M_1^{(r)} = \begin{Vmatrix} a_{11}^{(r)} & a_{12}^{(r)} & \cdots & a_{1r-1}^{(r)} & a_{1r+1}^{(r)} & \cdots & a_{1n}^{(r)} \\ a_{21}^{(r)} & a_{22}^{(r)} & \cdots & a_{2r-1}^{(r)} & a_{2r+1}^{(r)} & \cdots & a_{2n}^{(r)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m_r1}^{(r)} & a_{m_r2}^{(r)} & \cdots & a_{m_rr-1}^{(r)} & a_{m_rr+1}^{(r)} & \cdots & a_{m_rn}^{(r)} \end{Vmatrix}.$$

The n matrices $M_1^{(1)}, M_1^{(2)}, \cdots, M_1^{(n)}$ will be called the *I-minors* of $n - 1$ columns of the matrix M.

We may now form in a similar way the I-complement of any column

of the matrix $M_1^{(r)}$. The I-complement of the column headed by $a_{1s}^{(r)}$ will be denoted by $M_2^{(rs)}$. The $n(n-1)$ matrices $M_2^{(rs)}$, all possible choices of r and s being considered, will be called the I-minors of $n-2$ columns of the matrix M. More generally, we define recursively $M_{p+1}^{(r_1r_2\cdots r_{p+1})}$ as the I-complement of that column of $M_p^{(r_1r_2\cdots r_p)}$ which is headed by the element $a_{1r_{p+1}}^{(r_1r_2\cdots r_p)}$. We thus have relative to the matrix M a well-defined system of I-minors $M_p^{(r_1r_2\cdots r_p)}$, $(p = 1, 2, \cdots, n-1)$. The matrix M itself may be considered as the I-minor of n columns.* A noteworthy property of the system thus defined is that if $M_p^{(r_1r_2\cdots r_p)}$ is I-definite, then $M_{p+1}^{(r_1r_2\cdots r_{p+1})}$ is I-definite.

DEFINITION: *A matrix will be said to be of I-rank k if it possesses at least one I-minor of k columns which is I-definite, but does not possess any I-minor of $k+1$ columns which is I-definite. If none of its I-minors are I-definite it will be said to be of I-rank zero.*

As an immediate consequence of this definition there follows the

COROLLARY: *The I-rank of a matrix is not altered if*

(1) *any two rows or any two columns are interchanged;*

(2) *all elements of any row or any column are multiplied by the same positive constant.*

2. Systems of Linear Homogeneous Inequalities. Relative to the system of inequalities

$$
(1) \qquad
\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n > 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n > 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n > 0,
\end{array}
$$

the matrix of whose coefficients is M, we have the following theorems.

THEOREM I: *A necessary and sufficient condition for the existence of a solution of the system* (1) *is that the I-rank of the matrix M be greater than zero.*

THEOREM II: *If the I-rank of the matrix M is $k(>0)$, then the system* (1) *possesses a solution in which $k-1$ of the unknowns may be assigned values at pleasure.*

To prove Theorem II, suppose first that $k = n$. Then there is at least one column of M in which all elements have the same sign. We may suppose without loss of generality that the first column has this property, since in any case that condition could be brought about by a suitable ordering of the unknowns. The coefficients of x_1 in (1) are then all

* The total number of I-minors of M is easily seen to be $1 + \sum_{k=1}^{n-1} \frac{n!}{(n-p)!}$.

positive or all negative. In the former case the system is equivalent to

$$(2)\qquad \begin{array}{l} x_1 > -\dfrac{a_{12}}{a_{11}}x_2 - \dfrac{a_{13}}{a_{11}}x_3 - \cdots - \dfrac{a_{1n}}{a_{11}}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_1 > -\dfrac{a_{m2}}{a_{m1}}x_2 - \dfrac{a_{m3}}{a_{m1}}x_3 - \cdots - \dfrac{a_{mn}}{a_{m1}}x_n; \end{array}$$

and in the latter case to

$$(3)\qquad \begin{array}{l} x_1 < -\dfrac{a_{12}}{a_{11}}x_2 - \dfrac{a_{13}}{a_{11}}x_3 - \cdots - \dfrac{a_{1n}}{a_{11}}x_n \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_1 < -\dfrac{a_{m2}}{a_{m1}}x_2 - \dfrac{a_{m3}}{a_{m1}}x_3 - \cdots - \dfrac{a_{mn}}{a_{m1}}x_n. \end{array}$$

In either case $x_2, x_3, \cdots, x_n$ can be assigned values at pleasure. The variable x_1 is merely required in the former case to exceed the m lower bounds defined by (2) and in the latter case to be less than the m upper bounds defined by (3).

Next suppose $k < n$. By hypothesis there exists a k-columned I-minor of M which is I-definite. We may without loss of generality suppose that it is $M_{n-k}{}^{(12\cdots n-k)}$. We divide the inequalities of (1) into three classes according as the coefficients of x_1 are positive, negative, or zero, and as in §1, denote the leading coefficients of the three classes by a_{i1}, a_{j1}, a_{k1} respectively. Then the system (1) is equivalent to the system

$$(4_P)\qquad x_1 > -\frac{a_{i2}}{a_{i1}}x_2 - \frac{a_{i3}}{a_{i1}}x_3 - \cdots - \frac{a_{in}}{a_{i1}}x_n, \qquad i = i_1, i_2, \cdots, i_P;$$

$$(4_N)\qquad x_1 < -\frac{a_{j2}}{a_{j1}}x_2 - \frac{a_{j3}}{a_{j1}}x_3 - \cdots - \frac{a_{jn}}{a_{j1}}x_n, \qquad j = j_1, j_2, \cdots, j_N;$$

$$(4_Z)\qquad 0 > -a_{k2}x_2 - a_{k3}x_3 - \cdots - a_{kn}x_n, \qquad k = k_1, k_2, \cdots, k_Z.$$

By this system the variable x_1 is restricted to lie between the P lower bounds prescribed in (4_P) and the N upper bounds prescribed in (4_N). Possible values for x_1 will therefore exist if and only if the right side of each inequality of (4_P) is exceeded by the right side of every inequality of (4_N), that is if

$$(5)\qquad \left(\frac{a_{i2}}{a_{i1}} - \frac{a_{j2}}{a_{j1}}\right)x_2 + \left(\frac{a_{i3}}{a_{i1}} - \frac{a_{j3}}{a_{j1}}\right)x_3 + \cdots + \left(\frac{a_{in}}{a_{i1}} - \frac{a_{jn}}{a_{j1}}\right)x_n > 0,$$

$$i = i_1, i_2, \cdots, i_P,$$
$$j = j_1, j_2, \cdots, j_N.$$

We may then think of x_1 as having been *eliminated,* and proceed to a consideration of the conditions upon $x_2, x_3, \cdots, x_n$, imposed by (5) and (4_z).

Remembering that $-a_{i1}\cdot a_{j1} > 0$, we may write (5) in the equivalent form

$$(a_{i1}a_{j2} - a_{j1}a_{i2})x_2 + (a_{i1}a_{j3} - a_{j1}a_{i3})x_3 + \cdots + (a_{i1}a_{jn} - a_{j1}a_{in})x_n > 0.$$

These conditions, together with the conditions

$$a_{k2}x_2 + a_{k3}x_3 + \cdots + a_{kn}x_n > 0$$

imposed by (4_z), form a system

$$(6)\qquad \begin{array}{l} a_{i2}^{(1)}x_2 + a_{i3}^{(1)}x_3 + \cdots + a_{in}^{(1)}x_n > 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m_12}^{(1)}x_2 + a_{m_13}^{(1)}x_3 + \cdots + a_{m_1n}^{(1)}x_n > 0, \end{array}$$

whose matrix is what we have called the I-minor $M_1^{(1)}$.

If $M_1^{(1)}$ is not I-definite (that is if $k < n - 1$), we proceed to *eliminate* x_2 from the system (6) as x_1 was eliminated from (1), obtaining a system of inequalities in the variables $x_3, x_3, \cdots, x_n$, whose matrix is $M_2^{(12)}$.

By successive application of the process indicated, the variables $x_1, x_2, \cdots, x_{n-k}$, may be eliminated. The result will be the system

$$(7)\qquad \begin{array}{l} a_{1n-k+1}^{(v)}x_{n-k+1} + a_{1n-k+2}^{(v)}x_{n-k+2} + \cdots + a_{1n}^{(v)}x_n > 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m_vn-k+1}^{(v)}x_{n-k+1} + a_{m_vn-k+2}^{(v)}x_{n-k+2} + \cdots + a_{m_vn}^{(v)}x_n > 0 \end{array}$$

(v) represents the sequence $(12\cdots n-k)$,

whose matrix is $M_{n-k}^{(12\cdots n-k)}$ and is therefore I-definite. We may suppose that the column with respect to which this matrix is I-definite is the first. Hence the system (7) is equivalent to one of the two systems

$$(8)\qquad \begin{array}{l} x_{n-k+1} > -\dfrac{a_{1n-k+2}^{(v)}}{a_{1n-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{1n}^{(v)}}{a_{1n-k+1}^{(v)}}x_n \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ x_{n-k+1} > -\dfrac{a_{m_vn-k+2}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{m_vn}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_n, \end{array}$$

or

$$(9)\qquad \begin{array}{l} x_{n-k+1} < -\dfrac{a_{1n-k+2}^{(v)}}{a_{1n-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{1n}^{(v)}}{a_{1n-k+1}^{(v)}}x_n \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ x_{n-k+1} < -\dfrac{a_{m_vn-k+2}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_{n-k+2} - \cdots - \dfrac{a_{m_vn}^{(v)}}{a_{m_vn-k+1}^{(v)}}x_n. \end{array}$$

In either case, $x_{n-k+2}, x_{n-k+3}, \cdots, x_n$, can be assigned values at pleasure, and x_{n-k+1} will be limited by the lower bounds of (8) or the upper bounds of (9). The remaining variables $x_{n-k}, x_{n-k-1}, \cdots, x_1$ will have restricted but actually existent ranges defined by the systems of inequalities occurring in the elimination process, the system (4), defining the range for x_1 being typical. This completes the proof of Theorem II.

The *sufficiency* of the condition stated in Theorem I is implied by Theorem II. To prove the *necessity* of the condition, we note that the systems of inequalities presenting themselves in the successive steps of the process of elimination above described must necessarily be satisfied if a solution of (1) is to exist. If the I-rank of the matrix M is not greater than zero, the process of elimination can be continued until a system

$$a_{1n}{}^{(\mu)}x_n > 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a^{(\mu)}_{m_\mu n} > 0$$

containing only one variable is obtained. Since the matrix of this system, possessing as it does only one column, is by hypothesis not I-definite, the system is obviously inconsistent. This completes the proof of Theorem I.

3. Systems of Linear Non-homogeneous Inequalities. Consider the system of non-homogeneous inequalities

$$(10)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 &> 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 &> 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_m &> 0, \end{aligned}$$

the matrix of whose coefficients is M.

By a process of elimination similar to that used in the preceding section, we may obtain

THEOREM III: *If the I-rank of the matrix M is $k(>0)$, then the system (10) possesses a solution in which $k - 1$ of the unknowns may be assigned values at pleasure.*

From this theorem it follows that a sufficient condition for the existence of a solution of (10) is that the I-rank of M be greater than zero. That this condition is not necessary may be shown by a simple example. The system

$$x_1 + 1 > 0$$
$$-x_1 + 1 > 0$$

has a matrix of I-rank zero, but has the obvious solution

$$-1 < x_1 < 1.$$

A condition which is both necessary and sufficient for the existence of a solution of (10) may be obtained as follows. In (10) we make the substitution

$$x_1 = \frac{x_1'}{x_{n+1}'}, \qquad x_2 = \frac{x_2'}{x_{n+1}'}, \qquad \cdots, \qquad x_n = \frac{x_n'}{x_{n+1}'}$$

with the restriction

$$x_{n+1}' > 0. \tag{11}$$

Upon multiplication of each inequality of (10) by x_{n+1}', and annexation of (11), there results the system of homogeneous inequalities:

$$\begin{aligned} a_{11}x_1' + a_{12}x_2' + \cdots + a_{1n}x_n' + b_1x_{n+1}' &> 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ a_{m1}x_1' + a_{m2}x_2' + \cdots + a_{mn}x_n' + b_mx_{n+1}' &> 0 \\ x_{n+1}' &> 0. \end{aligned} \tag{12}$$

Every solution $(x_1', x_2', \cdots, x_n', x_{n+1}')$ of the system (12) affords a solution

$$(x_1, x_2, \cdots, x_n) = \left(\frac{x_1'}{x_{n+1}'}, \frac{x_2'}{x_{n+1}'}, \cdots, \frac{x_n'}{x_{n+1}'}\right)$$

of the system (10). And conversely, to every solution $(x_1, x_2, \cdots, x_n)$ of (10) there corresponds a solution

$$(x_1', x_2', \cdots, x_n', x_{n+1}') = (x_1, x_2, \cdots, x_n, 1)$$

of (12). But a necessary and sufficient condition for the existence of a solution of (12) is by Theorem I that the I-rank of the matrix

$$N = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\ 0 & 0 & \cdots & 0 & 1 \end{matrix} \right\|$$

be greater than zero. Hence

THEOREM IV: *A necessary and sufficient condition for the existence of a solution of the system* (10) *is that the I-rank of the matrix N be greater than zero.*

4. The Determination of the Solution. In the process of elimination carried out in §2, it was assumed that since the I-rank of the matrix M was k, the I-minor $M_{n-k}^{(12\cdots n-k)}$ was I-definite. The effect of this assumption was to assure us that upon the elimination of the variables $x_1, x_2, \cdots, x_{n-k}$, in their natural order, the matrix of the resulting system would be I-definite, and $k - 1$ of the unknowns could be assigned values at

pleasure. While this assumption was justifiable in the existence proof of §2, it must be recognized that for a given system of inequalities with numerical coefficients, the number of unknowns whose values are arbitrary in the solution depends upon the order in which the elimination is made.

Consider for example the simple system

$$x + y > 0$$

$$-x + y > 0.$$

If we solve first for x we obtain

$$-y < x < y$$

which implies the restriction $y > 0$.

But if we solve first for y, we obtain the solution

$$y > \begin{cases} x \\ -x \end{cases}, \qquad x \text{ arbitrary}$$

In the former case both variables are restricted; in the latter, one is arbitrary. Both solutions represent the totality of number pairs (x, y) which satisfy the given system of inequalities.*

If merely a determination of the totality of number sets $(x_1, x_2, \cdots, x_n)$ which satisfy a system of inequalities of form (1) is desired, it may be obtained by the process of elimination described in §2, the order of elimination being immaterial. The calculation required is merely the formation of a sequence of I-minors of the matrix of coefficients, each I-minor being the I-complement of a column of its predecessor, the process to be continued until an I-definite matrix is obtained, or until the sequence is automatically terminated by an I-minor of one column. The successive I-minors are the matrices of the successive systems of inequalities occurring in the elimination. If no one of them is I-definite, the given system admits no solution. If the last one is I-definite, the possible ranges for the variables can be determined from them by inspection.

Example: Solve the system

$$\begin{aligned} x_1 - 3x_2 + x_3 - x_4 &> 0 \\ 2x_1 - 3x_2 + x_3 + 2x_4 &> 0 \\ 3x_1 - 2x_2 + 3x_3 + x_4 &> 0 \\ -2x_1 + 2x_2 - x_3 + 2x_4 &> 0. \end{aligned}$$

* The difference in form of the solutions is immediately explained if the number pair (x, y) be interpreted as the coördinates of a variable point in a plane.

We form the sequence of I-minors

$$M = \begin{Vmatrix} 1 & -3 & 1 & -1 \\ 2 & -3 & 1 & 2 \\ 3 & -2 & 3 & 1 \\ -2 & 2 & -1 & 2 \end{Vmatrix}, \qquad M_1{}^{(1)} = \begin{Vmatrix} -4 & 1 & 0 \\ -2 & 0 & 8 \\ 2 & 3 & 8 \end{Vmatrix}, \qquad M_2{}^{(12)} = \begin{Vmatrix} 14 & 32 \\ 6 & 32 \end{Vmatrix}.$$

Since $M_2{}^{(12)}$ is I-definite with respect to both columns either x_3 or x_4 may be arbitrary, the other being restricted in terms of it. From $M_1{}^{(1)}$, x_2 may be bounded in terms of x_3 and x_4; and from M, x_1 may be bounded in terms of x_2, x_3 and x_4. A complete solution is

$$x_4 \text{ arbitrary}, \qquad x_3 > \begin{cases} -\frac{16}{7}x_4 \\ -\frac{16}{3}x_4, \end{cases}$$

$$\left.-\tfrac{3}{2}x_3 - 4x_4\right\} < x_2 < \begin{cases} \frac{1}{4}x_3 \\ 4x_4 \end{cases}$$

$$\left.\begin{aligned} 3x_2 - x_3 + x_4 \\ \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 - x_4 \\ -\tfrac{2}{3}x_2 - x_3 - x_4 \end{aligned}\right\} < x_1 < \left\{ x_2 - \tfrac{1}{2}x_3 + x_4. \right.$$

A solution in which *two* of the variables are arbitrary may be obtained by forming the I-minors

$$M = \begin{Vmatrix} 1 & -3 & 1 & -1 \\ 2 & -3 & 1 & 2 \\ 3 & -2 & 3 & 1 \\ -2 & 2 & -1 & 2 \end{Vmatrix}, \qquad M_1{}^{(3)} = \begin{Vmatrix} -1 & -1 & 1 \\ 0 & -1 & 4 \\ -3 & 4 & 7 \end{Vmatrix}.$$

$$x_1 \text{ and } x_2 \text{ arbitrary}, \qquad x_4 > \begin{cases} x_1 + x_2 \\ \frac{1}{4}x_2, \\ \frac{3}{7}x_1 - \frac{4}{7}x_2 \end{cases}$$

$$\left.\begin{aligned} -x_1 + 3x_2 + x_4 \\ -2x_1 + 3x_2 - 2x_4 \\ -x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_4 \end{aligned}\right\} < x_3 < \left\{ -2x_1 + 2x_2 + 2x_4. \right.$$

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ON THE SHORTEST LINE BETWEEN TWO POINTS IN NON-EUCLIDEAN GEOMETRY.

By T. H. Gronwall.

In his Science et hypothèse, Poincaré considers the geometry of a space interior to a sphere of radius R in which the line element equals $ds/(R^2 - r^2)$, where r is the distance from the center of the sphere and ds the ordinary euclidean line element. Poincaré states without proof (l. c., p. 65) that in this geometry the shortest line joining two given points is a circle through these points and orthogonal to the sphere. A simple proof of this theorem (which is of course well known in non-euclidean geometry) may be of some interest to the readers of the *Annals*.*

Using polar coördinates r, φ, θ, where the meridian plane $\varphi = 0$ passes through the two given points, we have $ds^2 = dr^2 + r^2 \sin^2 \theta d\varphi^2 + r^2 d\theta^2$, and the length of any curve $\varphi = \varphi(r)$, $\theta = \theta(r)$ is

$$\int \frac{\sqrt{1 + r^2 \sin^2 \theta \cdot \varphi'^2 + r^2\theta'^2}}{R^2 - r^2} dr,$$

the integral being taken between the limits r_1 and r_2 corresponding to the two given points. This integral exceeds or equals

$$\int \frac{\sqrt{1 + r^2\theta'^2}}{R^2 - r^2} dr \tag{1}$$

(which represents the length of the plane curve $\varphi = 0$, $\theta = \theta(r)$), equality taking place only when $\sin \theta \cdot \varphi' = 0$ for every point on the curve. Any point for which $\sin \theta = 0$ evidently lies in the meridian plane $\varphi = 0$, and $\varphi' = 0$ gives $\varphi = \text{const.} = 0$ (since $\varphi = 0$ at the two given points). Our shortest line is therefore a plane curve, and is found by minimizing the integral (1). The solution of the Euler-Lagrange equation

$$\frac{\partial F}{\partial \theta} - \frac{d}{dr}\frac{\partial F}{\partial \theta'} = 0 \tag{2}$$

(where $F(r, \theta, \theta')$ is the integrand in (1)) passing through the two given points makes (1) an *absolute* minimum,† and since in (1) F is independent

* In the American Math. Monthly, vol. 23 (1916), pp. 305–306, B. F. Finkel gives a proof in which, however, the shortest line is assumed from the outset to be a plane curve, and the differential equation of the problem is integrated in a rather complicated manner.

† Since (1) belongs to a general class of integrals shown to have this property by Carathéodory; see Bolza, Vorlesungen über Variationsrechnung, chapter IX.

of θ, (2) becomes $\frac{d}{dr}\frac{\partial F}{\partial \theta'} = 0$, whence $\frac{\partial F}{\partial \theta'} = $ const. or

$$\frac{r^2\theta'}{(R^2 - r^2)\sqrt{1 + r^2\theta'^2}} = \frac{1}{2\sqrt{a^2 - R^2}},$$

where a is a constant. Solving for θ',

$$\pm\,\theta' = \frac{R^2 - r^2}{r\sqrt{4a^2r^2 - (R^2 + r^2)^2}},$$

whence

$$\pm\, d\theta = \frac{\left(\frac{R^2}{r} - 1\right)dr}{\sqrt{4a^2 - \left(r + \frac{R^2}{r}\right)^2}} = -\frac{d\left(r + \frac{R^2}{r}\right)}{\sqrt{4a^2 - \left(r + \frac{R^2}{r}\right)^2}}$$

and integrating, α being an integration constant, we find

$$\pm\,(\theta - \alpha) = \text{arc cos}\,\frac{1}{2a}\left(r + \frac{R^2}{r}\right),$$

or

(3) $$r^2 - 2\,ar\cos(\theta - \alpha) + R^2 = 0.$$

Writing this equation $r^2 + a^2 - 2\,ar\cos(\theta - \alpha) = a^2 - R^2$, the cosine theorem shows that it represents a circle with center at $r = a$, $\theta = \alpha$ and radius $\sqrt{a^2 - R^2}$, and since $a^2 = R^2 + (\sqrt{a^2 - R^2})^2$, this circle intersects the circle $r = R$ (and consequently the sphere) orthogonally.

Let ψ be the angle between the lines joining the center of the orthogonal circle to a point on it and to the center of the sphere; then

$$r^2 = a^2 + (\sqrt{a^2 - R^2})^2 - 2a\sqrt{a^2 - R^2}\cos\psi$$

and $ds = \sqrt{a^2 - R^2}d\psi$. Moreover, let ψ_1 and ψ_2 be the ψ-values corresponding to the two given points, and ψ_0 and $-\psi_0$, where $\sin\psi_0 = R/a$, those corresponding to the intersection points of the orthogonal circle and the sphere. Then

$$\int \frac{ds}{R^2 - r^2} = \frac{1}{2a}\int_{\psi_1}^{\psi_2} \frac{d\psi}{\cos\psi_0 - \cos\psi} = \frac{1}{2R}\log\frac{\sin\frac{\psi_1 + \psi_0}{2}\sin\frac{\psi_2 - \psi_0}{2}}{\sin\frac{\psi_1 - \psi_0}{2}\sin\frac{\psi_2 + \psi_0}{2}},$$

an expression familiar in non-euclidean metrics.

THE GENERALIZED GAMMA FUNCTIONS.

By Emil L. Post.

Introduction.

The difference equation

$$\phi(z+1) = f(z)\phi(z) \tag{1}$$

has been studied directly* and indirectly† through

$$\phi(z+1) - \phi(z) = \psi(z)$$

in the cases where $f(z)$ is a meromorphic function. In the present paper a solution of (1) is obtained under an entirely different assumption with regard to $f(z)$. One class of functions satisfying this condition is of the form

$$g(z)h(z)e^{\psi(z)}$$

where $g(z)$ is regular at infinity, and $h(z)$ and $\psi(z)$ are any algebraic functions, Abelian integrals, or finite combinations of these. It is also attempted to bring out the similarity of the solution and its properties to those of the ordinary Gamma function.

In part I a solution of (1) is obtained as an infinite product. An asymptotic expression is obtained for it, as well as an infinite integral. In part II a number of relations are obtained of which the generalization of the multiplication theorem of the ordinary Gamma function is characteristic.

Part I: Fundamental Expansions.

1. Construction of the Gaussian Form of the Generalized Gamma Functions. Let $f(z)$ satisfy the following two conditions: (a) that $\log f(z)$ be analytic in a sector enclosing the positive end of the real axis; (b) that for some value of r, a positive real value of ϵ may be found such that

$$\lim_{p\to+\infty} p^{1+\epsilon}\frac{d^{r+1}}{dz^{r+1}}\log f(z+p) = 0$$

uniformly over any finite region of the z plane. Under these conditions a

* Mellin, Acta Math., vol. 8 (1886), pp. 37–80; Barnes, Proc. London Math. Soc., ser. 2, vol. 2 (1905), pp. 438–469.

† Guichard, Ann. de L'Ecole Norm., ser. 3, vol. 4 (1887), pp. 361–380; Appell, Journ. de Math., ser. 4, vol. 7 (1891), pp. 157–219; Hurwitz, Acta Math., vol. 20 (1896), pp. 285–312.

solution of

$$\phi(z+1) = f(z)\phi(z), \tag{1}$$

denoted by $\Gamma_{f(u)}(z)$, will be obtained which is entirely analogous to the ordinary Gamma function.

In analogy with

$$\Gamma(z) = \operatorname*{Lim}_{p\to\infty} \frac{1\cdot 2\cdots(p-1)}{z(z+1)\cdots(z+p-1)}\,p^z$$

set

$$\Gamma_{f(u)}(z) = \operatorname*{Lim}_{p\to\infty} \frac{f(1)f(2)\cdots f(p-1)}{f(z)f(z+1)\cdots f(z+p-1)}\,[f(p)]^z F(p,z) \tag{2}$$

where $F(p, z)$ is to be determined so that (2) converges, and satisfies the fundamental difference equation. For the latter condition

$$\operatorname*{Lim}_{p\to\infty} \frac{F(p, z+1)/F(p, z)}{f(z+p)/f(p)} = 1 \tag{3}$$

Taking into account (a) and (b), we may have condition (3) fulfilled by setting

$$\log F(p,z) = \frac{d}{dp}\log f(p)\,\frac{\phi_2(z)}{2!} + \cdots + \frac{d^r}{dp^r}\log f(p)\,\frac{\phi_{r+1}(z)}{(r+1)!} \tag{4}$$

where $\phi^2(z), \cdots \phi_{r+1}(z)$ are the Bernouillian polynomials.* Later we shall show that the same value of $F(p, z)$ insures the convergence of (2).

Since $\phi_n(1) = 0$,

$$\Gamma_{f(u)}(1) = 1$$

so that where q is a positive integer

$$\Gamma_{f(u)}(q) = f(1)f(2)\cdots f(q-1) = \left|\underline{f(q-1)}\right.$$

in an evident notation.

The general solution of (1) imay be written

$$\phi(z) = \Gamma_{f(u)}(z)P(z) \tag{5}$$

where $P(z)$ is any periodic function of period unity. Clearly $\Gamma_{f(u)}(z)$ is the only solution of (1) such that

$$\operatorname*{Lim}_{p\to\infty} \frac{\phi(z+p)}{\left|\underline{f(p-1)}\right.[f(p)]^z F(p,z)} = 1 \tag{6}$$

Equation (6) in connection with (1) may therefore be taken as defining $\Gamma_{f(u)}(z)$.

* Whittaker and Watson, Modern Analysis, Second edition, pp. 126, 127.

2. **Eulerian Form and Convergence.** Equation 2, §1 may be rewritten as follows:

$$\Gamma_{f(u)}(z) = \frac{[f(1)]^z F(1, z)}{f(z)} \prod_{p=1}^{\infty} \frac{f(p)}{f(z+p)} \left[\frac{f(p+1)}{f(p)}\right]^z \frac{F(p+1, z)}{F(p, z)} \tag{1}$$

$$= \frac{[f(1)]^z F(1, z)}{f(z)} \prod_{p=1}^{\infty} Q(p).$$

Since

$$\frac{\phi_2(z)}{(s-1)!\,2!} + \frac{\phi_3(z)}{(s-2)!\,3!} + \cdots + \frac{\phi_s(z)}{s!} = \frac{z^s - z}{s!},$$

we have

$$\log F(p+1, z) = \log F(p, z) + \sum_{s=1}^{r} \frac{z^s - z}{s!} \frac{d^s \log f(p)}{dp^s}$$

$$+ \sum_{s=1}^{r-1} \lambda_s \frac{d^{r+1} \log f(p+\theta_s)}{dp^{r+1}} \frac{\phi_{s+1}(z)}{(r-s+1)!\,(s+1)!}$$

$|\lambda_s| \leqq 1; \quad 0 < \theta_s < 1,$

by using Darboux's* form of the remainder in Taylor's series for complex variables in connection with (4 §1). Hence substituting in (1) and reducing we obtain

$$\log Q(p) = \sum_{s=1}^{r-1} \lambda_s \frac{d^{r+1} \log f(p+\theta_s)}{dp^{r+1}} \frac{\phi_{s+1}(z)}{(r-s+1)!\,(s+1)!}$$

$$+ \frac{z\lambda_r}{(r+1)!} \frac{d^{r+1}}{dp^{r+1}} \log f(p+\theta_r) - \frac{z^{r+1}\lambda_{r+1}}{(r+1)!} \frac{d^{r+1}}{dz^{r+1}} \log f(p+\theta_{r+1} z).$$

By means of condition (b) we see that from some value of (p) on, the terms of $\Sigma \log Q(p)$ are less in absolute value than those of the convergent series $\Sigma(1/(p^{1+\epsilon}))$. Hence (1) is absolutely convergent for all finite values of z which are not zeros of some $f(z+p)$. Condition (b) likewise proves (1) to be uniformly convergent over any finite region of the z plane which excludes these points. $\Gamma_{f(u)}(z)$ is therefore an analytic function, except for isolated points, in the entire sector at least over which $f(z)$ is analytic.

3. **Weierstrassian Form, and Derivatives.** When $r \leqq 1$,† equations (2 § 1) and (1 § 2) become

$$\Gamma_{f(u)}(z) = \lim_{p \to \infty} \frac{f(1)f(2)\cdots f(p-1)}{f(z)f(z+1)\cdots f(z+p-1)} [f(p)]^z, \tag{1}$$

and

$$\Gamma_{f(u)}(z) = \frac{[f(1)]^z}{f(z)} \prod_{p=1}^{\infty} \left\{ \frac{f(p)}{f(z+p)} \left[\frac{f(p+1)}{f(p)}\right]^z \right\}. \tag{2}$$

Since (1) is uniformly convergent, we may differentiate logarithmically so

* Journ. de Math., series 3, vol. 2 (1876), p. 291.

† Whenever r can be taken $\leqq 1$. This is not in general true.

that

$$\frac{\Gamma'_{f(u)}(z)}{\Gamma_{f(u)}(z)} = -\gamma_{f(u)} - \left[\left(\frac{f'(z)}{f(z)} - \frac{f'(1)}{f(1)}\right) + \left(\frac{f'(z+1)}{f(z+1)} - \frac{f'(2)}{f(2)}\right) + \cdots\right],$$

where

$$\gamma_{f(u)} = \lim_{p\to\infty}\left[\frac{f'(1)}{f(1)} + \frac{f'(2)}{f(2)} + \cdots + \frac{f'(p)}{f(p)} - \log f(p)\right] \tag{3}$$

convergence of $\gamma_{f(u)}$ being easily established. Clearly

$$\Gamma'_{f(u)}(1) = -\gamma_{f(u)}.$$

The analogy with $\Gamma(z)$ is further brought out by transforming (1) into

$$\frac{1}{\Gamma_{f(u)}(z)} = f(z)e^{\gamma_{f(u)}z}\prod_{p=1}^{\infty}\left\{\frac{f(p+z)}{f(p)}e^{-\frac{f'(p)}{f(p)}z}\right\}, \tag{4}$$

the generalization of the Weierstrassian form of $\Gamma(z)$.

From (3) we find

$$-\frac{d^2}{dz^2}\log\Gamma_{f(u)}(z) = \frac{d^2}{dz^2}\log f(z) + \frac{d^2}{dz^2}\log f(z+1) + \cdots.$$

More generally for r unrestricted, we have

$$-\frac{d^{r+1}}{dz^{r+1}}\log\Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots. \tag{5}$$

4. Asymptotic Expansions. In the notation we have adopted

$$\log \underline{|f(p)} = \log f(1) + \log f(2) + \cdots + \log f(p).$$

If $\phi(z)$ is analytic for $R(z) > a$, $-1 \leqq I(z) \leqq 1$, then

$$\phi(1) + \phi(2) + \cdots + \phi(p) = C + \int_a^p \phi(t)dt + \tfrac{1}{2}\phi(p) + \frac{B_1}{2!}\phi'(p)$$
$$- \frac{B_2}{4!}\phi'''(p) + \cdots + (-)^q\frac{B_q}{(2q)!}\phi^{(2q-1)}(p) + R_p,$$

where R_p can be put in either of the forms

$$\sum_{t=1}^{2q} A_t \sum_{s=p}^{\infty}\phi^{(2q+1)}(s + {}_t\theta_{s-p}), \qquad \sum_{t=1}^{2q+1} A_t \sum_{s=q}^{\infty}\phi^{(2q+2)}(s + {}_t\theta_{s-p}),$$

provided these forms converge. In the case where $\phi(z) = \log f(z)$, condition (a) insures the fulfilment of the condition of this formula. Let that form of the remainder be chosen which makes the index of differentiation $r + 1$. Then by condition (b) a value of p may be chosen such that

for all greater values

$$|R_p| < \frac{1}{p^{1+\epsilon}} + \frac{1}{(p+1)^{1+\epsilon}} + \cdots < \frac{1}{\epsilon(p-1)^\epsilon},$$

Hence $\lim\limits_{p\to\infty} R_p = 0$. Letting $C = \log G_{f(u)}$, we have

$$\begin{aligned}\log f(p) \sim \log G_{f(u)} + \int_a^p \log f(t)dt + \tfrac{1}{2}\log f(p)\\ + \frac{B_1}{2!}\frac{d}{dp}\log f(p) - \cdots + (-)^{q-1}\frac{B_q}{(2q)!}\frac{d^{2q-1}\log f(p)}{dp^{2q-1}}.\end{aligned} \tag{1}$$

In general q may be taken to be larger than the value above adopted, since condition (b) will usually be satisfied for larger values of r than the one used in the infinite product.

We shall now show that if for p we substitute z in the above formula, we obtain an asymptotic expansion for $\log \Gamma_{f(u)}(z+1)$, or

$$\begin{aligned}\log \Gamma_{f(u)}(z) \sim \log G_{f(u)} + \int_a^z \log f(t)dt - \tfrac{1}{2}\log f(z)\\ + \frac{B_1}{2!}\frac{d\log f(z)}{dz} - \cdots + \frac{(-)^{q-1}B_q}{(2q)!}\frac{d^{2q-1}\log f(z)}{dz^{2q-1}}.\end{aligned} \tag{2}$$

Denote the right hand member by S_z. Using the recurrence formulæ for the Bernoullian numbers, we easily find, provided z is within the analytic sector,

$$S_{z+1} = S_z + \log f(z) + P_z,$$

where P_z can be written in either of the forms

$$\sum_{t=1}^{q+1} A_t \frac{d^{2q+1}}{dz^{2q+1}}\log f(z+\theta_t), \qquad \sum_{t=1}^{q+2} A_t \frac{d^{2q+2}}{dz^{2q+2}}\log f(z+\theta_t).$$

Hence

$$S_{z+p} - S_z = \log[f(z)f(z+1)\cdots f(z+p-1)] + \sum_{y=z}^{z+p-1} P_y. \tag{3}$$

Substituting this result, and (1) above in the infinite product for $\Gamma_{f(u)}(z)$, we have

$$\begin{aligned}\log \Gamma_{f(u)}(z) = \lim_{p\to\infty}\Big[S_p + S_z - S_{z+p} + z\log f(p)\\ + \log F(p,z) + R_p - \sum_{y=z}^{z+p-1} P_y\Big].\end{aligned}$$

Now

$$S_{z+p} - S_p = \int_p^{z+p} \log f(t)dt - \tfrac{1}{2}\log\frac{f(z+p)}{f(p)} + \cdots$$

$$+\frac{(-)^{q-1}B_q}{(2q)!}\frac{d^{2q-1}}{dp^{2q-1}}\log\frac{f(z+p)}{f(p)}$$

$$= z\log f(p) + \frac{d}{dp}\log f(p)\left[\frac{z^2}{2!} - \frac{z}{2}\right] + \frac{d^2}{dp^2}\log f(p)\left[\frac{z^3}{3!} - \frac{z^2}{2\cdot 2!}\right.$$

$$\left.+\frac{B_1 z}{2!}\right] + \cdots + \frac{d^r}{dp^r}\log f(p)\left[\frac{z^{r+1}}{(r+1)!} - \frac{z^r}{2r!} + \frac{z^{r-1}B_1}{2!\,(r-1)!}\right.$$

$$\left.-\frac{z^{r-3}B_2}{4!\,(r-3)!} + \cdots\right] + Q_p$$

$$= z\log f(p) + \log F(p, z) + Q_p,$$

where

$$Q_p = \frac{z^{r+2}\lambda_1}{(r+2)!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_1 z) - \frac{1}{2}\frac{z^{r+1}\lambda_2}{(r+1)!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_2 z)$$

$$+\frac{B_1}{2!}\frac{z^r\lambda_3}{r!}\frac{d^{r+1}}{dp^{r+1}}\log f(p+\theta_3 z) - \cdots,$$

the last term ending in z^2 or z^3. Hence

$$\lim_{p\to\infty} Q_p = 0; \qquad \lim_{p\to\infty} R_p = 0; \qquad \lim_{p\to\infty}\sum_{y=z}^{z+p-1} P_y = \sum_{s=0}^{\infty} P_{z+s},$$

and

$$\log \Gamma_{f(u)}(z) = S_z - \sum_{s=0}^{\infty} P_{z+s},$$

S_z will therefore be the asymptotic expansion of $\log \Gamma_{f(u)}(z)$, provided z approaches infinity in such a way that $\sum_{s=0}^{\infty} P_{z+s} \to 0$. This will clearly be the case if $z \to \infty$ along a line parallel to or on the real axis in the positive direction. Under this condition (2) holds. We cannot infer more from the conditions we have assumed. If however condition (b) holds when $p \to \infty$ along any line,* (2) will hold for $z \to \infty$ along any line not parallel to or on the real axis in the negative direction.

5. **An Integral for $\Gamma_{f(u)}(z)$.** If x_1 and x_2 are integers, and $\phi(\xi)$ is a function which is analytic and bounded for all values of ξ such that

$$x_1 \leqq R(\xi) \leqq x_2,$$

then†

$$\tfrac{1}{2}\phi(x_1) + \phi(x_1+1) + \phi(x_1+2) + \cdots + \phi(x_2-1) + \tfrac{1}{2}\phi(x_2)$$

$$= \int_{x_1}^{x_2}\phi(\xi)d\xi + \frac{1}{i}\int_0^\infty \frac{\phi(x_2+iy) - \phi(x_1+iy) - \phi(x_2-iy) + \phi(x_1-iy)}{e^{2\pi y}-1}dy.$$

* This extended condition is satisfied by the class of functions suggested in the introduction.

† Whittaker and Watson, p. 145, Ex. 7.

Let the initial conditions (*a*) and (*b*) imposed on $f(z)$ be extended to the following:

(a') $\log f(z)$ is analytic to the right of a line at distance a from the axis of imaginaries;

(b') for some value of r and ϵ, $\epsilon > 0$,

$$\lim_{z\to\infty} z^{1+\epsilon}\frac{d^{r+1}}{dz^{r+1}}\log f(z) = 0,$$

where $z \to \infty$ along any line included in the analytic region of (a').

We shall then have, if $R(z) > a$, $x_1 = 0$, $x_2 \to \infty$,

$$-\frac{d^{r+1}}{dz^{r+1}}\log\Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots$$

$$= \frac{1}{2}\frac{d^{r+1}}{dz^{r+1}}\log f(z) + \int_0^\infty \frac{d^{r+1}}{d\xi^{r+1}}\log f(z+\xi)d\xi$$

$$-\frac{1}{i}\int_0^{\infty}\frac{\frac{d^{r+1}}{dz^{r+1}}\log f(z+iy) - \frac{d^{r+1}}{dz^{r+1}}\log f(z-iy)}{e^{2\pi y}-1}dy + \lim_{x_2\to\infty} Rx_2,$$

where

$$Rx_2 = \frac{1}{i}\int_0^{\infty}\frac{\frac{d^{r+1}}{dz^{r+1}}\log f(z+x_2+iy) - \frac{d^{r+1}}{dz^{r+1}}\log f(z+x_2-iy)}{e^{2\pi y}-1}dy.$$

From (b') we easily find

$$\lim_{x_2\to\infty} x_2^{\epsilon} Rx_2 = 0, \quad i.\ e., \lim_{x_2\to\infty} Rx_2 = 0.$$

Also

$$\int_0^\infty \frac{d^{r+1}}{d\xi^{r+1}}\log f(z+\xi)d\xi = -\frac{d^r\log f(z)}{dz^r},$$

so that on integration

$$\log\Gamma_{f(u)}(z) = c_0 + c_1 z + \cdots + c_r z^r - \frac{1}{2}\log f(z)$$

$$+\int_a^z \log f(z)dz + \frac{1}{i}\int_0^\infty \frac{\log f(z+iy) - \log f(z-iy)}{e^{2\pi y}-1}dy.$$

If we let $z \to \infty$, on expanding the latter integral and comparing with the asymptotic expansion, we find

$$c_0 = \log G_{f(u)}; \quad c_1 = c_2 = \cdots c_r = 0,$$

and

$$\log\Gamma_{f(u)}(z) = \log G_{f(u)} - \frac{1}{2}\log f(z) + \int_0^z \log f(z)dz$$

$$+2\int_0^\infty \frac{\log f(z+iy) - \log f(z-iy)}{2i}\,\frac{dy}{e^{2\pi y}-1}.$$

As a matter of notation, let

$$\frac{\phi(z+iy)-\phi(z-iy)}{2i}=\sin_{\phi(u)}(z,y),$$

$$\frac{\phi(z+iy)+\phi(z-iy)}{2}=\cos_{\phi(u)}(z,y).$$

Then

$$\log\Gamma_{f(u)}(z)=\log G_{f(u)}-\frac{1}{2}\log f(z)+\int_a^z\log f(z)dz+2\int_0^\infty\frac{\sin_{\log f(u)}(z,y)}{e^{2\pi y}-1}dy \quad (1)$$

and by differentiation

$$\frac{\Gamma_{f(u)}'(z)}{\Gamma_{f(u)}(z)}=\log f(z)-\frac{1}{2}\frac{f'(z)}{f(z)}+2\int_0^\infty\frac{\cos_{f'(u)/f(u)}(z,y)}{e^{2\pi y}-1}dy. \quad (2)$$

Part II. Transformations.

6. **Integral for the Asymptotic Constant.** In the present section and in the one following we shall obtain results which are very useful in establishing particular relations between the generalized Gamma functions.

Let*

$$u=\int_z^{z+1}\log\Gamma_{f(u)}(t)dt;$$

then

$$\frac{du}{dz}=\log\Gamma_{f(u)}(z+1)-\log\Gamma_{f(u)}(z)=\log f(z),$$

and

$$u=\int_a^z\log f(t)dt+C.$$

Since

$$\log\Gamma_{f(u)}(z)\sim\log G_{f(u)}+\int_a^z\log f(t)dt-\frac{1}{2}\log f(z)+\sum_{s=1}^{q}\frac{(-)^{s-1}B_s}{(2s)!}\frac{d^{2s-1}\log f(z)}{dz^{2s-1}},$$

it easily follows that

$$\int_z^{z+1}\log\Gamma_{f(u)}(t)dt\sim\log G_{f(u)}+\int_z^{z+1}\left[\int_a^t\log f(u)du\right]dt$$
$$-\frac{1}{2}\int_z^{z+1}\log f(t)dt+\sum_{s=1}^{q}\frac{(-)^{s-1}B_s}{(2s)!}\frac{d^{2s-2}}{dz^{2s-2}}\log\frac{f(z+1)}{f(z)}.$$

But, using the Euler-Maclaurin Sum formula,† we get

* For the analogue of the ordinary Gamma function see Whittaker and Watson, p. 255, Ex. 21.

† Whittaker and Watson, p. 128.

$$\int_z^{z+1}\left[\int_a^t \log f(u)du\right]dt \sim \int_a^z \log f(t)dt$$

$$+\frac{1}{2}\int_z^{z+1}\log f(t)dt + \sum_{s=1}^{q}\frac{(-)^s B_s}{(2s)!}\frac{d^{2s-2}}{dz^{2s-2}}\log\frac{f(z+1)}{f(z)},$$

so that

$$\int_z^{z+1}\log \Gamma_{f(u)}(t)dt \sim \int_a^z \log f(t)dt + \log G_{f(u)}.$$

Comparing with the above, we see that

$$C = \log G_{f(u)}$$

and

$$\int_z^{z+1}\log \Gamma_{f(u)}(t)dt = \int_a^z \log f(t)dt + \log G_{f(u}. \tag{1}$$

Letting $z = a$, we obtain

$$\log G_{f(u)} = \int_a^{a+1}\log \Gamma_{f(u)}(t)dt, \tag{2}$$

an analytical expression for the constant $G_{f(u)}$.

Since $G_{f(u)}$ depends on the value of a chosen, we shall write it ${}_aG_{f(u)}$. Then

$$\log_{a_1} G_{f(u)} = \log_{a_2} G_{f(u)} + \int_{a_2}^{a_1}\log f(t)dt. \tag{3}$$

7. **The Asymptotic Test.** We have seen that $\log \Gamma_{f(u)}(z)$ has the same asymptotic expansion as $\log |f(p-1)|$. Since any other solution than $\Gamma_{f(u)}(z)$ of

$$\phi(z+1) = f(z)\phi(z) \tag{1}$$

must be in the form

$$\phi(z) = \Gamma_{f(u)}(z)P(z), \tag{2}$$

where $P(z)$ is periodic of period unity, it is evident that $\Gamma_{f(u)}(z)$ is the only solution of (1) possessing this property. The following is a more useful expression of the above principle.

From

$$-\frac{d^{r+1}}{dz^{r+1}}\log \Gamma_{f(u)}(z) = \frac{d^{r+1}}{dz^{r+1}}\log f(z) + \frac{d^{r+1}}{dz^{r+1}}\log f(z+1) + \cdots,$$

we have

$$\operatorname*{Lim}_{z\to+\infty}\frac{d^{r+1}}{dz^{r+1}}\log \Gamma_{f(u)}(z) = 0. \tag{3}$$

Let $\psi(z)$ be some other solution of (1) possessing property (3). Then if $z = p + x$, where p is an integer, we must have for all values of x

$$\operatorname*{Lim}_{p\to\infty}\frac{d^{r+1}}{dx^{r+1}}\log \psi(x+p) = \operatorname*{Lim}_{p\to\infty}\left[\frac{d^{r+1}}{dx^{r+1}}\log \Gamma_{f(u)}(x+p) + \frac{d^{r+1}}{dx^{r+1}}\log P(x)\right] = 0,$$

so that

$$\frac{d^{r+1}}{dx^{r+1}} \log P(x) = 0.$$

Since $P(z)$ is periodic we can only have

$$P(z) = ce^{2\pi ipz},$$

so that

$$\psi(z) = ce^{2\pi ipz}\Gamma_{f(u)}(z), \tag{4}$$

where p is an integer. If furthermore $\psi(z)$ have an asymptotic expansion of the form

$$b + a_0 \int_a^z \log f(t)dt + a_1 \log f(z) + \cdots + a_{2q} \frac{d^{2q-1}}{dz^{2q-1}} \log f(z),$$

where the a's are independent of $f(z)$, we find by letting $f(z) = e^{z^{r-\epsilon}}$ and using (4) that

$$a_0 = 1, \quad a_1 = -\tfrac{1}{2}, \qquad a_{2s+1} = 0, \qquad a_{2s} = \frac{B_s}{(2s)!}(-)^{s-1},$$

and

$$\psi(z) = c\Gamma_{f(u)}(z). \tag{5}$$

We shall refer to condition (3) and the one just given as the asymptotic test. Hence if two solutions $\psi_1(z)$ and $\psi_2(z)$ of (1) satisfy the asymptotic test,

$$\psi_1(z) = c\psi_2(z).$$

The same is clearly true of solutions of

$$\phi(z + n) = f(z)\phi(z),$$

where n is real and positive, since this equation can be transformed into

$$\psi(z + 1) = f(nz)\psi(z).$$

8. **Elementary Transformations.** From the Gaussian form of $\Gamma_{f(u)}(z)$ we see that

$$\Gamma_{[f_1(u)]^m[f_2(u)]^n}(z) = [\Gamma_{f_1(u)}(z)]^m[\Gamma_{f_2(u)}(z)]^n. \tag{1}$$

In particular

$$\Gamma_c(z) = c^{z-1}; \qquad \Gamma_u(z) = \Gamma(z); \qquad \Gamma_{e^{(u^r)}}(z) = e^{[\phi_{r+1}(z)]/(r+1)}.^*$$

Again, both $\Gamma_{f(u+b)}(z)$ and $\Gamma_{f(u)}(z + b)$ are solutions of

$$\phi(z + 1) = f(z + b)\phi(z).$$

They clearly have the same asymptotic expansions, except for a constant

* By means of these results and (2), we easily evaluate $\Gamma_{f_1(u)e^{f_2(u)}}(z)$ where $f_1(u)$ is a rational and $f_2(u)$ an integral algebraic function.

factor, so that

$$\Gamma_{f(u+b)}(z) = c\Gamma_{f(u)}(z+b).$$

To determine c, let $z = 1$. Since $\Gamma_{f(u+b)}(1) = 1$,

$$c = \frac{1}{\Gamma_{f(u)}(1+b)},$$

and

$$\Gamma_{f(u+b)}(z) = \frac{\Gamma_{f(u)}(z+b)}{\Gamma_{f(u)}(1+b)}. \tag{2}$$

By means of the integral of §6 these results may be directly applied to ${}_aG_{f(u)}$. We thus find

$${}_aG_{[f_1(u)]^m[f_2(u)]^n} = [{}_aG_{f_1(u)}]^m[{}_aG_{f_2(u)}]^n,$$

$${}_oG_c = c^{-1/2}, \qquad {}_oG_{e^{(u^r)}} = 1,$$

$${}_aG_{f(u+b)} = \frac{{}_{a+b}G_{f(u)}}{\Gamma_{f(u)}(1+b)}, \tag{3}$$

9. **Infinite Products in Terms of Generalized Gamma Functions.** Consider first

$$\prod_{p=0}^{\infty}\frac{f(a_1+p)f(a_2+p)\cdots f(a_k+p)}{f(b_1+p)f(b_2+p)\cdots f(b_l+p)} = \prod_{p=0}^{\infty}F(p). \tag{1$'$}$$

A sufficient condition for convergence is that for some positive value of ϵ

$$\operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}\log F(p) = 0. \tag{2$'$}$$

But

$$\begin{aligned}\log F(p) &= \log f(a_1+p) + \cdots - \log f(b_1+p) - \cdots \\ &= (k-l)\log f(p) + \frac{\Sigma a - \Sigma b}{1}\frac{d}{dp}\log f(p) + \cdots \\ &\qquad\qquad + \frac{\Sigma a^r - \Sigma b^r}{r!}\frac{d^r}{dp^r}\log f(p) + R_p,\end{aligned}$$

where

$$\operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}R_p = 0, \qquad \operatorname*{Lim}_{p\to\infty} p^{1+\epsilon}\frac{d^r}{dp^r}\log f(p) \neq 0.$$

We must therefore have

$$k = l, \qquad \Sigma a = \Sigma b, \cdots \Sigma a^r = \Sigma b^r, \tag{3$'$}$$

if (2)′ is to be fulfilled. Now

$$\prod_{p=0}^{\infty}\frac{f(a_1+p)\cdots f(a_k+p)}{f(b_1+p)\cdots f(b_k+p)} = \frac{\Gamma_{f(u)}(b_1)\cdots\Gamma_{f(u)}(b_k)}{\Gamma_{f(u)}(a_1)\cdots\Gamma_{f(u)}(a_k)}S_\infty,$$

where

$$S_p = \frac{[f(p)]^{a_1}F(p, a_1) \cdots [f(p)]^{a_k}F(p, a_k)}{[f(p)]^{b_1}F(p, b_1) \cdots [f(p)]^{b_k}F(p, b_k)}.$$

Using (3)′, we see that $S_p = 1$, so that

$$\prod_{p=0}^{\infty}\frac{f(a_1 + p)f(a_2 + p) \cdots f(a_k + p)}{f(b_1 + p)f(b_2 + p) \cdots f(b_k + p)} = \frac{\Gamma_{f(u)}(b_1) \cdots \Gamma_{f(u)}(b_k)}{\Gamma_{f(u)}(a_1) \cdots \Gamma_{f(u)}(a_k)}. \tag{1}$$

Consider now more generally

$$\prod_{p=0}^{\infty}\frac{f_1(a_1 + p) \cdots f_k(a_k + p)}{\phi_1(b_1 + p) \cdots \phi_l(b_l + p)} = \prod_{p=0}^{\infty} F(p), \tag{4'}$$

where (2)′ is again a sufficient condition. We may write

$$\begin{aligned}\prod_{p=0}^{\infty} F(p) &= \operatorname*{Lim}_{q\to\infty}\prod_{p=1}^{q} F(p-1) = \operatorname*{Lim}_{q\to\infty} \underline{|F(q-1)}\\ &= {}_aG_{F(u-1)}e^{\int_a^\infty \log F(u-1)du}\end{aligned} \tag{5'}$$

by (1 §4). Furthermore, provided the functions used exist, from §8 it follows that

$$\begin{aligned}{}_aG_{F(u-1)} &= \frac{{}_aG_{f_1(a_1-1+u)} \cdots {}_aG_{f_k(a_k-1+u)}}{{}_aG_{\phi_1(b_1 1-1+u)} \cdots {}_aG_{\phi_l(b_l-1+u)}}\\ &= \frac{{}_{a+a_1-1}G_{f_1(u)} \cdots {}_{a+a_k-1}G_{f_k(u)}}{{}_{a+b_1-1}G_{\phi_1(u)} \cdots {}_{a+b_l-1}G_{\phi_l(u)}}\,\frac{\Gamma_{\phi_1(u)}(b_1) \cdots \Gamma_{\phi_l(u)}(b_l)}{\Gamma_{f_1(u)}(a_1) \cdots \Gamma_{f_k(u)}(a_k)},\end{aligned}$$

so that finally

$$\prod_{p=0}^{\infty}\frac{f_1(a_1 + p) \cdots f_k(a_k + p)}{\phi_1(b_1 + p) \cdots \phi_l(b_l + p)} = A\,\frac{{}_aG_{f_1(u)} \cdots {}_aG_{f_k(u)}}{{}_aG_{\phi_1(u)} \cdots {}_aG_{\phi_l(u)}}\,\frac{\Gamma_{\phi_1(u)}(b_1) \cdots \Gamma_{\phi_l(u)}(b_l)}{\Gamma_{f_1(u)}(a_1) \cdots \Gamma_{f_k(u)}(a_k)},$$

where

$$\begin{aligned}\log A = \int_a^\infty \log F(u-1)du &+ \sum_{\lambda=1}^{k}\int_a^{a+a_\lambda-1}\log f_\lambda(u)du\\ &- \sum_{\mu=1}^{l}\int_a^{a+b_\mu-1}\log\phi_\mu(u)du.\end{aligned} \tag{2}$$

10. **The Multiplication Theorem.** Both $\Gamma_{f(u/n)}(nz)$ and

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z + \frac{1}{n}\right) \cdots \Gamma_{f(u)}\left(z + \frac{n-1}{n}\right)$$

are solutions of

$$\phi(z+1) = f(z)f\left(z + \frac{1}{n}\right) \cdots f\left(z + \frac{n-1}{n}\right)\phi(z).$$

Furthermore it is easily shown that they both satisfy the asymptotic test

since $\Gamma_{f(u)}(z)$ does. Hence

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)=\psi(n)\Gamma_{f(u/n)}(nz).$$

To determine $\psi(n)$, we have

$$\log {}_{na}G_{f(u/u)} = \int_{na}^{na+1}\log\Gamma_{f(u/u)}(z)dz = n\int_a^{a+(1/n)}\log\Gamma_{f(u/u)}(nz)dz$$

$$= n\int_a^{a+(1/n)}\left[\log\Gamma_{f(u)}(z)+\cdots+\log\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)-\log\psi(n)\right]dz$$

$$= n\left[\int_a^{a+(1/n)}\log\Gamma_{f(u)}(z)dz+\int_{a+(1/n)}^{a+(2/n)}\log\Gamma_{f(u)}(z)dz+\cdots\right.$$
$$\left.+\int_{a+[(n-1)/n]}^{a+1}\log\Gamma_{f(u)}(z)dz\right]-\log\psi(n)$$

$$= n\int_a^{a+1}\log\Gamma_{f(u)}(z)dz-\log\psi(n),$$

so that

$$\psi(n)=\frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/u)}},$$

and

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right)=\frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/u)}}\Gamma_{f(u/n)}(nz) \qquad (1)$$

When $f(u) = u$, by using the results of §8, the ordinary multiplication theorem results.

Let $z = 1/n$, and we obtain

$$\Gamma_{f(u)}\left(\frac{1}{n}\right)\Gamma_{f(u)}\left(\frac{2}{n}\right)\cdots\Gamma_{f(u)}\left(\frac{n-1}{n}\right)=\frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}. \qquad (2)$$

If in this $n = 2$, we obtain

$$\Gamma_{f(u)}\left(\frac{1}{2}\right)=\frac{[{}_aG_{f(u)}]^2}{{}_{2a}G_{f(u/2)}}, \qquad (3)$$

the analogue of $\Gamma(1/2) = \sqrt{\pi}$.

11. **An Integration Theorem Generalized.** When n is a positive integer

$$\int_a^{a+1}\log f(z)dz=\int_a^{a+(1/n)}\log f(z)dz+\int_{a+(1/n)}^{a+(2/n)}\log f(z)dz+\cdots$$
$$+\int_{a+[(n-1)/n]}^{a+1}\log f(z)dz.$$

By means of the Generalized Gamma functions the corresponding relation

may be obtained for any positive real value of n;*

$$\int_a^{a+(1/n)} \log f(z)dz + \cdots + \int_{a+(n-1/n)}^{a+1} \log f(z)dz$$

$$= \int_a^{a+(1/n)} \log \left[f(z)f\left(z+\frac{1}{n}\right)\cdots f\left(z+\frac{n-1}{n}\right)\right]dz$$

$$= \int_a^{a+(1/n)} \log [f(z)\Gamma_{f[z+(u/n)]}(n)]dz.$$

Let now n have any positive real value. Then

$$\int_a^{a+(1/n)} \log [f(z)\Gamma_{f[z+(u/n)]}(n)]dz = \int_a^{a+(1/n)} \log \frac{\Gamma_{f[u/n]}(n+nz)}{\Gamma_{f[u/n]}(nz)}\,dz\dagger$$

$$= \frac{1}{n}\left[\int_{na+n}^{na+n+1} \log \Gamma_{f[u/n]}(z)dz - \int_{na}^{na+1} \log \Gamma_{f[u/n]}(z)dz\right],$$

so that, using §6, we find the relation desired

$$\int_a^{a+(1/n)} \log [f(z)\Gamma_{f[z+(u/n)]}(n)]dz = \int_a^{a+1} \log f(z)dz. \tag{1}$$

12. **The Multiplication Theorem Generalized.** If condition (b) be extended so that for some value of r

$$\operatorname*{Lim}_{p\to\infty} p^{2+\epsilon} \frac{d^{r+2}\log f(z+p)}{dz^{r+2}} = 0,$$

the multiplication theorem may be generalized to admit all positive real values of n.‡ We have for positive integral values of n

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right) = \Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n).$$

By the condition above imposed we have

$$\operatorname*{Lim}_{p\to\infty} p^{1+\epsilon} \frac{d^{r+2}}{d\zeta^{r+2}} \log \Gamma_{f(u)}\left(z+\frac{\zeta+p}{n}\right) = 0.$$

It therefore follows easily that $\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)$ exists when n is any real and positive number. Furthermore, with the aid of (2 §8)

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)$$

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* The theorem depends only on the existence of the functions involved. That n be real and positive is sufficient for this purpose, under conditions (a) and (b), but not always necessary. For the class of functions suggested in the introduction the theorem holds for all values of n.

† By (2 § 8) and (1 § 1).

‡ The note to §11 applies here too except that n may not be a negative real number.

is seen to satisfy the equation

$$\phi\left(z+\frac{1}{n}\right)=f(z)\phi(z).$$

Since after transformation by (2 §8) it may be shown to satisfy the asymptotic test, we obtain

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n) = \psi(n)\Gamma_{f(u/n)}(nz).$$

The previous section enables us to determine $\psi(n)$ as in the simpler case.

$$\begin{aligned}\log {}_{na}G_{f(u/n)} &= n\int_a^{a+(1/n)} \log \Gamma_{f(u/n)}(nz)dz \\ &= n\int_a^{a+(1/n)} [\log \Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n)]dz - \log \psi(n) \\ &= n\int_a^{a+1} \log \Gamma_{f(u)}(z)dz - \log \psi(n).\end{aligned}$$

Hence, as before

$$\psi(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}},$$

and

$$\Gamma_{f(u)}(z)\Gamma_{\Gamma_{f(u)}(z+(v/n))}(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}\Gamma_{f(u/n)}(nz) \tag{1}$$

Letting $z = 1/n$, and using (2 §8), we obtain

$$\Gamma_{\Gamma_{f(u)}(v/n)}(n) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}. \tag{2}$$

13. **The Associated Periodic Functions.** Let $f(z)$ be such that both $\Gamma_{f(u)}(z)$ and $\Gamma_{f(-u)}(z)$ exist, and let

$$\Gamma_{f(u)}(z)\Gamma_{f(-u)}(1-z) = F(z),$$

Then

$$F(z+1) = F(z).$$

We shall denote this periodic function by $P_{f(u)}(z)$, *i. e.*,

$$\Gamma_{f(u)}(z)\Gamma_{f(-u)}(1-z) = P_{f(u)}(z). \tag{1}$$

Let ${}_aG_{f(u)-a}G_{f(-u)} = {}_a\pi_{f(u)}$. Then using the integral of §6, we easily obtain

$$\log {}_a\pi_{f(u)} = \int_a^{a+1} \log P_{f(u)}(z)dz. \tag{2}$$

Clearly

$$P_{f(u)}(0) = \frac{1}{f(0)}, \qquad \text{or} \qquad \lim_{z\to 0} P_{f(u)}(z)f(z) = 1,$$

$$P_{f(u)}(\tfrac{1}{2}) = \Gamma_{f(u)}(\tfrac{1}{2})\Gamma_{f(-u)}(\tfrac{1}{2}) = \frac{[{}_a\pi_{f(u)}]^2}{{}_{2a}\pi_{f(u/2)}}. \tag{3}$$

Again

$$P_{f(u)}(z) = P_{f(-u)}(-z), \tag{4}$$

$$P_{f(u+b)}(z) = \frac{1}{f(b)}\frac{P_{f(u)}(z+b)}{P_{f(u)}(1+b)}, \tag{5}$$

The multiplication theorem gives

$$\Gamma_{f(u)}(z)\Gamma_{f(u)}\left(z+\frac{1}{n}\right)\cdots\Gamma_{f(u)}\left(z+\frac{n-1}{n}\right) = \frac{[{}_aG_{f(u)}]^n}{{}_{na}G_{f(u/n)}}\Gamma_{f(u/n)}(nz),$$

$$\Gamma_{f(-u)}\left(\frac{1}{n}-z\right)\Gamma_{f(-u)}\left(\frac{2}{n}-z\right)\cdots\Gamma_{f(-u)}(1-z) = \frac{[{}_{-a}G_{f(-u)}]^n}{{}_{-na}G_{f(-u/n)}}\Gamma_{f(-u/n)}(1-nz).$$

Inverting and multiplying,* we finally obtain

$$P_{f(u)}(z)P_{f(u)}\left(z+\frac{1}{n}\right)\cdots P_{f(u)}\left(z+\frac{n-1}{n}\right) = \frac{[{}_a\pi_{f(u)}]^n}{{}_{na}\pi_{f(u/n)}}P_{f(u/n)}(nz). \tag{6}$$

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* It will be noticed that whereas in the case of the ordinary Γ function it is usual to obtain the multiplication theorem from that of the sine function, the reverse method is used here.

ON THE MOST GENERAL PLANE CLOSED POINT-SET THROUGH WHICH IT IS POSSIBLE TO PASS A SIMPLE CONTINUOUS ARC.*

By R. L. Moore and J. R. Kline.

A set of points is said to be totally disconnected if it contains no connected subset consisting of more than one point. In 1905 L. Zoretti† showed that every closed, bounded and totally disconnected set of points is a subset of a Cantorean‡ line. In 1906 F. Riesz§ attempted to show that every such set of points is a subset of a simple continuous arc.|| Shortly thereafter Zoretti¶ pointed out that Riesz's argument was fallacious. He, however, left unsettled the question whether Riesz's theorem was true or false. In 1910, in an article that contains no reference either to Riesz or to Zoretti, Denjoy** indicated that this theorem could be proved with the use of certain ideas contained in a former paper†† of his own. We have not, however, succeeded in determining from his meager indications just what sort of argument he had in mind. At any rate, in order that a closed and bounded point-set should be a subset of a simple continuous arc it is of course not *necessary* that it should be totally disconnected. In the present paper we will establish the following result.

Theorem 1. *In order that a closed and bounded point-set M should be a subset of a simple continuous arc it is necessary and sufficient that every closed, connected subset of M should be either a single point or a simple continuous arc t such that no point of t, with the exception of its endpoints is a limit point of $M - t$.*‡‡

* Presented to the American Mathematical Society, February 24, 1917.

† Sur les fonctions analytiques uniformes, Journal de Mathematiques pures et appliquées, vol. 1 (1905), p. 12.

‡ A Cantorean line is a closed connected point-set that contains the interior of no circle.

§ Sur les ensembles discontinus, Comptes Rendus, vol. 141 (1905), pp. 650–655.

|| It is well known that not every Cantorean line is a simple continuous arc.

¶ Sur les ensembles discontinus, Comptes Rendus, vol. 142 (1906), pp. 763–764. Riesz made use of the false proposition that the orthogonal projection of a closed totally disconnected point-set is itself necessarily disconnected.

** Continu et discontinu, Comptes Rendus, vol. 151 (1910), pp. 138–140.

†† Sur les ensembles parfaits discontinus, Comptes Rendus, vol. 149 (1909), pp. 1048–1050.

‡‡ Consider the point-set $\overline{M}$ (Fig. 1) composed of the straight interval t from (0, 0) to (2, 0) together with the infinite set of points (1, 1), (1, 1/2), (1, 1/3), (1, 1/4), $\cdots$. Every closed connected subset of $\overline{M}$ is either a single point or a simple continuous arc. But the point (1, 0) of

In our proof of this theorem we will make use of the following lemmas.

LEMMA 1. *If G is a finite set of simple closed curves and M is a closed point-set and each point of M is within some curve of the set G_1 but no point of M is on any curve of the set G, then there exists a finite set $\bar{G}$ of simple closed curves such that* 1) *each point of M is within some curve of $\bar{G}$,* 2) *every curve of the set $\bar{G}$ lies entirely without every other curve of the set $\bar{G}$,* 3) *each curve of the set of $\bar{G}$ is within some curve of the set G_1.*

Proof. If P is a point of M there exists* a closed curve $\bar{J}_P$ such that (1) every point of $\bar{J}_P$ is on some curve of the set G, (2) every point within $\bar{J}_P$ is within every curve of the set G that encloses P and without every curve of G that does not enclose P. The set M_P of all those points of M that lie within $\bar{J}_P$ is closed. It follows† that there exists a closed curve J_P that lies within $\bar{J}_P$ and encloses M_P. Such a curve will be said to *properly* enclose P. It is clear that if J_{P_1} and J_{P_2} are curves that properly enclose the points P_1 and P_2 respectively, then J_{P_1} either lies entirely without J_{P_2} or properly encloses every point of M that is enclosed by J_{P_2}. For each point P of M select just one J_P and let K denote the set of curves thus obtained. By the Heine-Borel Theorem there is a finite subset $\bar{G}$ of the set of curves K such that every point of M is within some curve of the set $\bar{G}$. The set $\bar{G}$ satisfies the conditions of Lemma 1.

LEMMA 2. *Suppose that M is a closed and bounded set of points and G is a set of closed curves such that* (1) *every point of M is either on or within*

the arc t, though not an endpoint of t, is a limit point of the set of points $M - t$. It is accordingly impossible to pass a simple continuous arc through $\bar{M}$.

FIG. 1.

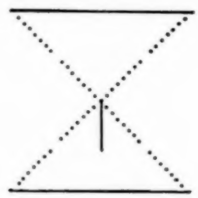

FIG. 2.

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ON THE MOST GENERAL PLANE CLOSED POINT-SET THROUGH WHICH IT IS POSSIBLE TO PASS A SIMPLE CONTINUOUS ARC.*

By R. L. Moore and J. R. Kline.

A set of points is said to be totally disconnected if it contains no connected subset consisting of more than one point. In 1905 L. Zoretti† showed that every closed, bounded and totally disconnected set of points is a subset of a Cantorean‡ line. In 1906 F. Riesz§ attempted to show that every such set of points is a subset of a simple continuous arc.‖ Shortly thereafter Zoretti¶ pointed out that Riesz's argument was fallacious. He, however, left unsettled the question whether Riesz's theorem was true or false. In 1910, in an article that contains no reference either to Riesz or to Zoretti, Denjoy** indicated that this theorem could be proved with the use of certain ideas contained in a former paper†† of his own. We have not, however, succeeded in determining from his meager indications just what sort of argument he had in mind. At any rate, in order that a closed and bounded point-set should be a subset of a simple continuous arc it is of course not *necessary* that it should be totally disconnected. In the present paper we will establish the following result.

Theorem 1. *In order that a closed and bounded point-set M should be a subset of a simple continuous arc it is necessary and sufficient that every closed, connected subset of M should be either a single point or a simple continuous arc t such that no point of t, with the exception of its endpoints is a limit point of $M - t$.*‡‡

* Presented to the American Mathematical Society, February 24, 1917.

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‡ A Cantorean line is a closed connected point-set that contains the interior of no circle.

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‡‡ Consider the point-set $\overline{M}$ (Fig. 1) composed of the straight interval t from (0, 0) to (2, 0) together with the infinite set of points (1, 1), (1, 1/2), (1, 1/3), (1, 1/4), ⋯. Every closed connected subset of $\overline{M}$ is either a single point or a simple continuous arc. But the point (1, 0) of

In our proof of this theorem we will make use of the following lemmas.

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Proof. If P is a point of M there exists* a closed curve $\overline{J}_P$ such that (1) every point of $\overline{J}_P$ is on some curve of the set G, (2) every point within $\overline{J}_P$ is within every curve of the set G that encloses P and without every curve of G that does not enclose P. The set M_P of all those points of M that lie within $\overline{J}_P$ is closed. It follows† that there exists a closed curve J_P that lies within $\overline{J}_P$ and encloses M_P. Such a curve will be said to *properly* enclose P. It is clear that if J_{P_1} and J_{P_2} are curves that properly enclose the points P_1 and P_2 respectively, then J_{P_1} either lies entirely without J_{P_2} or properly encloses every point of M that is enclosed by J_{P_2}. For each point P of M select just one J_P and let K denote the set of curves thus obtained. By the Heine-Borel Theorem there is a finite subset $\overline{G}$ of the set of curves K such that every point of M is within some curve of the set $\overline{G}$. The set $\overline{G}$ satisfies the conditions of Lemma 1.

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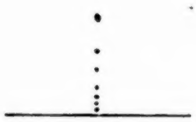

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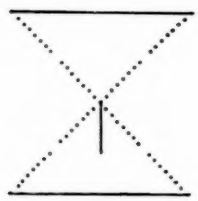

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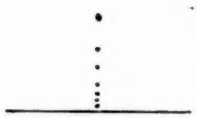

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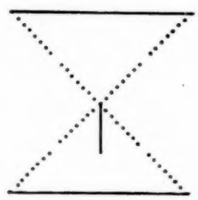

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some curve of G, (2) if a point of M is not within any curve of G then there exists a curve g of the set G such that P is on g and such that, if P is a limit point of any subset of M, then every such subset of M contains points within g. Then there is a finite set of curves $\overline{G}$ such that every curve of $\overline{G}$ is a curve of G and such that $\overline{G}$ satisfies with respect to M the same conditions that are assumed above as being satisfied by G.

Proof. If P is a point of M that does not lie within any curve of the set G there exists a closed curve J_P belonging to G and containing P such that P is not a limit point of any point-set that contains no points within J_P. There exists a closed curve C_P enclosing P such that every point of M within C_P, except the point P, is within J_P. Let K denote the set of all such curves C_P for all such points P. Every point of M is within a curve of the set $G + K$. By the Heine Borel Theorem there exists a finite set of curves C_{P_1}, C_{P_2}, C_{P_3}, $\cdots$, C_{P_n}, belonging to K, and a finite set g_1, g_2, g_3, $\cdots$, g_m, belonging to G, such that every point of M is within a curve of one or the other of these two sets. It is clear that the set $\overline{G}$ of curves $g_1, g_2, g_3, \cdots, g_m, J_{P_1}, J_{P_2}, J_{P_3}, \cdots, J_{P_n}$ satisfies the requirements of Lemma 2.

Proof of Theorem 1. In order that a closed and bounded set of points should be a subset of a simple continuous arc it is evidently *necessary* that it should satisfy the conditions imposed in the statement of Theorem 1. We will proceed to show that these conditions are also *sufficient*. Suppose that M is a point-set that fulfills all of these conditions. An arc will be called a *maximum arc* if it belongs to M but is not a subset of any other arc that belongs to M. If A and B are the endpoints of such a maximum arc AB, it can be proved with the use of methods employed in the proof of Theorem 32 of Foundations that there exist two arcs AXB and AYB, with no point in common except the points A and B, such that the closed curve $AXBYA$ formed by these arcs encloses every point of the arc AB except the points A and B but encloses no point of M that is not on the arc AB. By a theorem due to Zoretti* there exists a closed curve J_A that encloses A but not B, lies within a circle of radius 1 and does not contain B or any point of M that is not on the arc AB. By Theorem 43 of Foundations there exists a simple closed curve a containing A such that every point of a belongs either to $AXBYA$ or to J_A and such that every point within a is without $AXBYA$ and within J_A. The curve a contains the point A but no other point of M, encloses no point of AB and lies within a circle of radius 1. Similarly there exists a closed curve b that lies entirely without a, contains B but no other point of M and lies within a circle of radius 1. Each point P of M that is not a proper part of a connected

* Sur les fonctions analytiques uniformes, loc. cit., pp. 9–11.

subset of M can be enclosed by a simple closed curve p that lies within a circle of radius 1 and has no point in common with M. The point-set composed of all endpoints of maximum arcs, together with all those points of M that are not proper parts of connected subsets of M, is a closed set of points. It follows by Lemma 2 that there exists a finite set $\bar{G}$ of closed curves $\bar{a}_1, \bar{a}_2, \bar{a}_3, \cdots, \bar{a}_{\bar{n}}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \cdots, \bar{b}_{\bar{n}}, \bar{p}_1, \bar{p}_2, \bar{p}_3, \cdots, \bar{p}_{\bar{m}}$ and a set $\bar{\tau}$ of maximum arcs $\bar{A}_1\bar{B}_1, \bar{A}_2\bar{B}_2, \cdots, \bar{A}_n\bar{B}_n$ such that every point of M is either within a curve of the set $\bar{G}$ or on an arc of the set $\bar{\tau}$ and such that for every i $(1 \leqq i \leqq \bar{n})$ and j $(1 \leqq j \leqq \bar{m})$ (1) each of the curves $\bar{a}_i$, $\bar{b}_i$, $\bar{p}_j$ lies within a circle of radius 1, (2) $\bar{a}_i$ contains $\bar{A}_i$ and $\bar{b}_i$ contains $\bar{B}_i$ but neither of them contains any other point of M, (3) $\bar{a}_i$ lies entirely without $\bar{b}_i$, (4) the arc $\bar{A}_i\bar{B}_i$ lies entirely without $\bar{a}_i$ and entirely without $\bar{b}_i$ except that $\bar{A}_i$ and $\bar{B}_i$ are on $\bar{a}_i$ and $\bar{b}_i$ respectively, (5) $\bar{A}_i\bar{B}_i$ is a maximum arc, (6) $\bar{p}_j$ contains no point of M.

For each i $(1 \leqq i \leqq \bar{n})$ there exists, with center at $\bar{A}_i$ and with radius less than 1, a circle $\bar{C}_i$ which neither encloses nor contains a point of any curve of the set $\bar{G}$ except the curve $\bar{a}_i$, and which does not enclose every point of $\bar{a}_i$. Let O_i denote a point of $\bar{a}_i$ that is not within $\bar{C}_i$ and let $\bar{E}_i$ and $\bar{F}_i$ denote points in the order $O_i\bar{E}_i\bar{A}_i\bar{F}_i$ on $\bar{a}_i$. By Zoretti's theorem* there exists within $\bar{C}_i$ a closed curve $\bar{J}_i$ that encloses $\bar{A}_i$ and contains no point of $M - \bar{A}_i\bar{B}_i$. There exists an interval $\bar{X}_i\bar{Y}_i\bar{Z}_i$ of $\bar{J}_i$ that lies entirely within $\bar{a}_i$ except that its endpoints $\bar{X}_i$ and $\bar{Z}_i$ are on the segments $O_i\bar{E}_i\bar{A}_i$ and $O_i\bar{F}_i\bar{A}_i$ respectively of $\bar{a}_i$. The arc $\bar{X}_i\bar{Y}_i\bar{Z}_i$ and the interval $\bar{X}_i\bar{A}_i\bar{Z}_i$ of the closed curve $\bar{a}_i$ form a closed curve a_i' that neither encloses nor contains any point of any curve of the set $\bar{G}$ except $\bar{a}_i$ and has on it no point of M except the point $\bar{A}_i$. There exists an arc $X_i'Y_i'Z_i'$ lying entirely within $\bar{a}_i$ except that its endpoints X_i' and Z_i' are on $\bar{a}_i$ in the order $X_i'\bar{X}_i\bar{A}_i\bar{Z}_iZ_i'$ and such that (1) $X_i'Y_i'Z_i'$ contains no point of M or of any curve of the set $\bar{G}$, (2) every point of M that is within $\bar{a}_i$ is either within a_i' or within the closed curve a_i'' formed by $X_i'Y_i'Z_i'$ and that interval of $\bar{a}_i$ whose endpoints are X_i' and Z_i' and which does not contain $\bar{A}_i$. Similarly, for each i there exist two closed curves b_i' and b_i'' such that (1) every point of M that lies within $\bar{b}_i$ is within either b_i' or b_i'', (2) the curve b_i' contains $\bar{B}_i$ but no other point of M, (3) the curve b_i'' contains no point of M, (4) no curve, except $\bar{b}_i$, of the set $\bar{G}$ contains any point on or within b_i', (5) the interiors of b_i' and b_i'' are subsets of the interior of $\bar{b}_i$. No one of the set $\bar{G}'$ of curves $a_1', a_2', \cdots, a_{\bar{n}}', b_1', b_2', \cdots, b_{\bar{n}}'$ contains or encloses any point of any other curve of the set $\bar{G}'$ or of the set G'' of curves $a_1'', a_2'', \cdots, a_{\bar{n}}'', b_1'', b_2'', \cdots, b_{\bar{n}}'', \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{\bar{m}}$. If a curve of G'' encloses a point of $\bar{a}_i$ or of $\bar{b}_i$ it must enclose $\bar{a}_i$, $\bar{b}_i$ and $\bar{A}_i\bar{B}_i$. The set

* Loc. cit., pp. 9–11.

$\bar{G}'$ has as a subset a set G' of closed curves a_{m_1}', a_{m_2}', $\cdots$, a_{m_n}', b_{m_1}', b_{m_2}', $\cdots$, b_{m_n}' such that every point of M is either within some curve of G' or G'' or on some arc of the set $\bar{A}_{m_1}\bar{B}_{m_1}$, $\bar{A}_{m_2}\bar{B}_{m_2}$, $\cdots$, $\bar{A}_{m_n}\bar{B}_{m_n}$ and such that no curve of the set G'' either contains or encloses a point of any curve of the set G'. No curve of the set G'' contains a point of M. It follows with the aid of Lemma 1 that there exists a finite set of closed curves p_1, p_2, $\cdots$, p_m such that (1) every point of M that is within a curve of the set G'' is also within a curve of the set p_1, p_2, $\cdots$, p_m, (2) every curve of the set p_1, p_2, $\cdots$, p_m is without every other curve of this set and within some curve of the set G'' and therefore without every curve of the set G'. Replace the symbols a_{m_i}' by a_i, b_{m_i}' by b_i, $\bar{A}_{m_i}$ by A_i and $\bar{B}_{m_i}$ by B_i. We now have a set G of closed curves a_1, a_2, $\cdots$, a_n, b_1, b_2, $\cdots$, b_n, p_1, p_2, $\cdots$, p_m and a set Q of arcs A_1B_1, A_2B_2, $\cdots$, A_nB_n such that (1) every point of M is either within a curve of the set G or on an arc of the set Q, (2) no two arcs of the set Q have a point in common, (3) for every i ($1 \leqq i \leqq n$) the arc A_iB_i lies entirely without every curve of the set G except that A_i is on a_i and B_i is on b_i, (4) each curve of the set G is within a circle of radius 1, (5) every curve of the set G is entirely without every other curve of the set G. There exists a set $\bar{Q}$ of arcs C_1D_2, C_2D_3, C_3D_4, $\cdots$, $C_{n-1}D_n$, C_nE_1, F_1E_2, F_2E_3, F_3E_4 $\cdots$, $F_{m-1}E_m$ such that (1) $C_i (1 \leqq i \leqq n)$ is a point of b_i distinct from B_i, D_i ($2 \leqq i \leqq n$) is a point of a_i distinct from A_i, E_i ($1 \leqq i \leqq m$) is a point of p_i, F_i ($1 \leqq i \leqq m - 1$) is a point of p_i distinct from E_i, (2) no arc of the set $\bar{Q}$ has a point in common with any other arc of the set $\bar{Q}$ or any arc of the set Q, (3) every arc of $\bar{Q}$ lies, except for its endpoints, entirely without every curve of the set G. The arcs of Q and of $\bar{Q}$ and the curves of G form an arc-curve chain K_1* covering M. For every $i (1 \leqq i \leqq n)$ there exists an arc-curve chain $\left\{{\beta_i \atop \alpha_i}\right\}$ such that (1) $\left\{{\beta_i \atop \alpha_i}\right\}$ covers the set of all those points of M that lie within $\left\{{b_i \atop a_i}\right\}$, (2) the $\left\{{\text{first} \atop \text{last}}\right\}$ curve of $\left\{{\beta_i \atop \alpha_i}\right\}$ contains the point $\left\{{B_i \atop A_i}\right\}$ but lies except for this point entirely within $\left\{{b_i \atop a_i}\right\}$, (3) every arc of $\left\{{\beta_i \atop \alpha_i}\right\}$, and every curve of $\left\{{\beta_i \atop \alpha_i}\right\}$ except the $\left\{{\text{first} \atop \text{last}}\right\}$ one, lies entirely within $\left\{{b_i \atop a_i}\right\}$, (4) every curve of $\left\{{\beta_i \atop \alpha_i}\right\}$ is within some circle of radius 1/2. There exists within p_i ($1 \leqq i \leqq m$) an arc-curve chain ρ_i covering all those points of M that are within p_i and such that each curve of ρ_i is within some circle of radius 1/2. For every i ($2 \leqq i \leqq n$) there exists on the first curve of α_i a point X_i not lying on the first arc of α_i. There exists an arc X_iD_i that lies, except for the point D_i, entirely within a_i and has no point except X_i in common with any arc or curve of α_i. For every i

* An arc-curve chain is a finite set K of closed curves a_1, a_2, $\cdots$ a_n and arcs $A_1'A_2$, $A_2'A_3$, $A_3'A_4$, $\cdots$ $A_{n-1}'A_n$, such that (1) every curve of K is without every other curve of K, (2) no two arcs of K have a point in common, (3) for every i ($1 \leqq i \leqq n - 1$) the arc $A_i'A_{i+1}$ lies entirely without every curve of K except that A_i' is on a_i and A_{i+1} is on a_{i+1}. The arc-curve chain K is said to cover the point-set M if every point of M is either within a curve or on an arc of K.

$(1 \leqq i \leqq n)$ there exists, on the last curve of β_i, a point Y_i not lying on the last arc of β_i. There exists an arc Y_iC_i that lies, except for the point C_i, entirely within b_i and has no point except Y_i in common with any arc or curve of β_i. For every i $(1 \leqq i \leqq m)$ there exists, on the first curve of ρ_i, a point W_i that does not lie on the first arc of ρ_i. There exists an arc W_iE_i that lies entirely within p_i except for the point E_i and has no point, except W_i, in common with any arc or curve of ρ_i. For every i $(1 \leqq i \leqq m - 1)$ there exists, on the last curve of ρ_i, a point Z_i that does not lie on the last arc of ρ_i. There exists an arc Z_iF_i that lies entirely within p_i, except for the point F_i, and has no point in common with the arc W_iE_i or any arc of ρ_i and no point, except Z_i, in common with any curve of ρ_i. We now have a new chain K_2 whose curves are the curves of the chains $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n, \rho_1, \rho_2, \cdots, \rho_m$ and whose arcs are the arcs of these chains together with the arcs $X_2D_2, X_3D_3, \cdots, X_nD_n, Y_1C_1, Y_2C_2, \cdots, Y_nC_n, W_1E_1, W_2E_2, \cdots, W_mE_m, Z_1F_1, Z_2F_2, \cdots, Z_{m-1}F_{m-1}$. Every point of M is on an arc or within a curve of K_1 and also on an arc or within a curve of K_2. Every curve of K_1 is within a circle of radius 1. Every curve of K_2 is within a circle of radius 1/2. Every point of every curve of K_2 is within or on a curve of K_1 and every arc of K_2 is either within a curve or on an arc of K_1. There exists a chain K_3 each of whose curves is within a circle of radius 1/3 and which has a relation to K_3 similar to the above described relation of K_2 to K_1. This process may be continued. Thus there exists an infinite sequence of arc-curve chains $K_1, K_2, K_3, \cdots$ such that, for every n, (1) each point of M is on an arc or within a curve of K_n, (2) each curve of K_n is within some circle of radius $1/n$, (3) every point of each curve of K_{n+1} is within or on some curve of K_n and each arc of K_{n+1} is either within a curve or on an arc of K_n. Let $\overline{K}_n$ denote the set of all points $[X]$ such that X is on an arc or within a curve of K_n. Let t denote the set of all points that are common to $\overline{K}_1, \overline{K}_2, \overline{K}_3, \cdots$. It is clear that t contains every point of M. Let a_{n1} and b_{n1} respectively denote the first and last curves of the chain K_n. The interiors of the closed curves $a_{11}, a_{21}, a_{31}, \cdots$ have in common only one point A and those of the curves $b_{11}, b_{21}, b_{31}, \cdots$ have in common only one point B. That t is a simple continuous arc from A to B may be proved with the use of methods similar to those employed in the proof of Theorem 15 of Foundations.

UNIVERSITY OF PENNSYLVANIA.

REPEATED INTEGRALS.

By D. C. Gillespie.

Du Bois Reymond* proved that, when the bounded function $f(x, y)$ has a Riemann double integral over the fundamental rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, then the two repeated integrals exist and are each equal to the double integral. This theorem permits the explanation that the function $f(x, y)$ is an integrable function of x for each y is not implied in the statement that the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ exists. For obviously the existence of the double integral over the rectangle could not depend on the values of the function along a single line parallel to the x axis. The repeated integral with respect to x with respect to y is said to exist when the repeated upper integral and the repeated lower integral in this same order are equal to each other; *i. e.*,

$$\overline{\int_c^d} dy \overline{\int_a^b} dx = \underline{\int_c^d} dy \underline{\int_a^b} dx.$$

To this theorem of Du Bois Reymond there has now been added a corresponding theorem for Lebesgue integrals. It follows then that if the function $f(x, y)$ possesses a Lebesgue double integral and the repeated Riemann integrals exist they are equal to the Lebesgue double integral and hence to each other.†

W. H. Young, with no hypothesis concerning the existence of the double integral, proved that if the function $f(x, y)$ is an integrable function of x for each y and an integrable function of y for each x, all being Riemann integrals, then both repeated integrals exist.‡ Somewhat later, Lichtenstein by a different method obtained the same result and in addition proved the two repeated integrals were equal.§ In a later paper he extends his results to integrals taken over point sets not comprising all points of the fundamental rectangle.||

* Crelle's Journal, vol. 94 (1883), p. 277.

† Hobson, Theory of Functions of a Real Variable, p. 581.

‡ Monatshefte für Mathematik und Physik, vol. 2 (1910), p. 127. The argument seems to depend on the plane set of points at which $f(x, y)$ is discontinuous with respect to y being measurable.

§ Göttingen Nachrichten, (1910), pp. 468–475.

|| Sitzungsbericht der Berliner Mathematischen Gesellschaft, (1910–11), pp. 55–69.

These interesting papers just miss showing that if the two repeated integrals exist they are equal to each other. A proof of this theorem is given in this note.

If the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ exists, then the values of y for which

$$\left(\bar{\varphi}(y) \equiv \overline{\int_a^b} f(x, y)dx\right) > \left(\underline{\varphi}(y) \equiv \underline{\int_a^b} f(x, y)dx)\right)$$

form a set of measure zero. For the fact that

$$\bar{\varphi}(y) \geqq \underline{\varphi}(y) \text{ and that } \overline{\int_c^d} \bar{\varphi}(y)dy = \underline{\int_c^d} \underline{\varphi}(y)dy$$

shows that

$$\overline{\int_c^d} \bar{\varphi}(y)dy = \underline{\int_c^d} \bar{\varphi}(y)dy,$$

i. e., $\bar{\varphi}(y)$ is an integrable function of y. In the same way one sees that $\underline{\varphi}(y)$ is integrable. Then since $\int_c^d (\bar{\varphi}(y) - \underline{\varphi}(y))dy = 0$ and since the integrand is positive or zero its integral over any sub-interval of (c, d) is also zero. The function $(\bar{\varphi}(y) - \underline{\varphi}(y))$ is therefore an integrable null function* and hence different from zero at a set of measure zero.†

The values of y for which the integral $\int_a^b f(x, y)dx$ exists, being identical with those which satisfy the equation $\bar{\varphi}(y) = \underline{\varphi}(y)$, form a set everywhere dense. Moreover, as both repeated integrals are assumed to exist, the set of values of x, for which

$$\left(\bar{\psi}(x) \equiv \overline{\int_c^d} f(x, y)dy\right) > \left(\underline{\psi}(x) \equiv \underline{\int_c^d} f(x, y)dy\right),$$

also has the measure zero. We shall designate this set by G.

Let us now divide the interval from c to d into n equal parts and write the equation

$$\text{(1)} \qquad \frac{d-c}{n}[f(x, \eta_1) + f(x, \eta_2) + \cdots + f(x, \eta_n)] = \bar{\psi}(x) + R_n(x),$$

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* Hobson, loc. cit., pp. 347 and 348.

† While this condition is necessary for the existence of the repeated $\int_c^d dy \int_a^b f(x, y)dx$ it is of course not sufficient. Example: $f(x, y) = 0$ for x and y both rational, $f(x, y) = 1$ for other values of x and y. The fact that the existence of the repeated integral $\int_c^d dy \int_a^b f(x, y)dx$ does require that the integral $\int_a^b f(x, y)dx$ exist except for a set of values of y of measure zero could, I think, be combined with the results obtained by Lichtenstein to prove our theorem.

where η_i is a point of the ith subdivision for which $f(x, \eta_i)$ is an integrable function of x, and $R_n(x)$ is defined by the equation. The function $R_n(x)$ being the difference between two integrable functions of x is itself integrable, hence integrating both sides of equation (1) with respect to x there results

$$(2) \quad \frac{d-c}{n}[\varphi(\eta_1) + \varphi(\eta_2) + \cdots + \varphi(\eta_n)] = \int_a^b \bar{\psi}(x)dx + \int_a^b R_n(x)dx.$$

The set $G_{n,\epsilon}$ is defined to consist of those values of x which satisfy the inequality $|R_n(x)| > \epsilon$, where ϵ is a positive number. The function $R_n(x)$ is integrable; $|R_n(x)|$ is therefore also integrable and $G_{n,\epsilon}$ is measurable. Suppose n is allowed to vary, becoming infinite, there is then defined an infinite sequence of sets $G_{1,\epsilon}, G_{2,\epsilon}, G_{3,\epsilon} \cdots$ etc., of measure $m(G_{1,\epsilon}), m(G_{2,\epsilon}), m(G_{3,\epsilon}) \cdots$ etc. respectively. Now the $\operatorname{limit}_{n=\infty} m(G_{n,\epsilon}) = 0$. For if this were not true, an infinite number of these sets would each have measure greater than some positive number c. This, in turn, would require that there exist a set of measure greater than or equal to c each point of which belongs to an infinite number of the defined sets.* This is impossible, since only points of G can belong to an infinite number of the sets.

Both $\bar{\psi}(x)$ and $\frac{d-c}{n}[f(x, \eta_1) + f(x, \eta_2) + \cdots + f(x, \eta_n)]$ are less than or at most equal in absolute value to $u(d-c)$, where u is the least upper bound of $|f(x, y)|$ in the region $a \leqq x \leqq b$, $c \leqq y \leqq d$. It follows from equation (1) that $|R_n(x)| \leqq 2u(d-c)$ and as a consequence $\int_a^b |R_n(x)|dx \leqq m(G_{n,\epsilon})2u(d-c) + (b-a)\epsilon$. Then, finally, since $m(G_{n,\epsilon})$ approaches zero as n becomes infinite, $\operatorname{limit}_{n=\infty} \int_a^b R_n(x)dx = 0$. From equation (2) we obtain

$$\operatorname{limit}_{n=\infty} \frac{d-c}{n}[\varphi(\eta_1) + \varphi(\eta_2) + \cdots + \varphi(\eta_n)] = \int_a^b \bar{\psi}(x)dx + \operatorname{limit}_{n=\infty} \int_a^b R_n(x)dx,$$

or

$$\int_c^d dy \int_a^b f(x, y)dx = \int_a^b dx \int_c^d f(x, y)dy.$$

This theorem may now be extended to a certain class of unbounded functions.

We assume:

1°. $f(x, y) \geqq 0$;†

* Hobson, loc. cit., p. 120, § 93.

† Instead of 1° it is sufficient to assume that the function $f(x, y)$ is not both positively and negatively unbounded.

2°. All integrals are proper integrals, i. e., $f(x, y)$ is a bounded function of y for each x and a bounded function of x for each y; moreover the functions $\bar{\varphi}(y)$ and $\bar{\psi}(x)$ are bounded;

3°. The two repeated integrals exist.

The argument is a repretition of that for bounded functions up to where the inequality $|R_n(x)| \leqq 2u(d - c)$ is obtained. This inequality is meaningless when $f(x, y)$ is unbounded and the proof must be concluded without using it.

From 3° it follows that the limit as n becomes infinite of the left side of equation (2) exists. This fact establishes the existence of the

$$\lim_{n \doteq \infty} \int_a^b R_n(x)dx.$$

The function $f(x, y)$ being positive or zero, $\bar{\psi}(x)$ is also positive or zero. Since then $\bar{\psi}(x)$ is by hypothesis bounded, $R_n(x) \geqq -P$, where P is the least upper bound of $\bar{\psi}(x)$. Let ϵ be any positive number, $G_{n,-\epsilon}$ the set of values of x for which $R_n(x) \leqq -\epsilon$, and $m(G_{n,-\epsilon})$ the measure of this set. Now $\lim_{n \doteq \infty} m(G_{n,-\epsilon}) = 0$; for $G_{n,-\epsilon}$ is only a part of the set $|R_n(x)| \geqq \epsilon$ and, as has been shown, the measure of this second set approaches zero as n becomes infinite. It follows now from

$$\int_a^b R_n(x)dx \geqq m(G_{n,-\epsilon})(-P) + (b-a)(-\epsilon),$$

that the $\lim_{n \doteq \infty} \int_a^b R_n(x)dx \geqq 0$. This fact being established the result of passing to the limit in equation (2) is

$$\int_a^b dx \int_c^d f(x, y)dy \leqq \int_c^d dy \int_a^b f(x, y)dx.$$

An interchange of the rôles played by x and y in the argument would yield

$$\int_c^d dy \int_a^b f(x, y)dx \leqq \int_a^b dx \int_c^d f(x, y)dy.$$

The two repeated integrals are therefore equal.

If we retain conditions 2° and 3° but allow the function $f(x, y)$ to be both positively and negatively unbounded the theorem is no longer true.

Example:

$$f(0, 0) = 0, f(x, y) = \frac{2(x^3y - xy^3)}{(x^2 + y^2)^3}$$

for other values of x and y.

$$\int_0^1 f(x, y)dy = \frac{xy^2}{(x^2+y^2)^2}\Big]_0^1 = \frac{x}{(x^2+1)^2}.$$

$$\int_0^1 \frac{x}{(x^2+1)^2}dx = -\frac{1}{2(1+x^2)}\Big]_0^1 = \frac{1}{4},$$

whereas

$$\int_0^1 f(x_1y)dx = -\frac{x^2y}{(x^2+y^2)^2}\Big]_0^1 = \frac{-y}{(1+y^2)^2}$$

and

$$\int_0^1 \frac{-y}{(1+y^2)^2}dy = \frac{1}{2(1+y^2)}\Big]_0^1 = -\frac{1}{4}.$$

CORNELL UNIVERSITY.

RELATIONS BETWEEN ABSTRACT GROUP PROPERTIES AND SUBSTITUTION GROUPS.

By G. A. Miller.

It is customary to exhibit some of the fundamental relations existing between abstract groups and substitution groups by means of the following rectangular arrangement of the operators of the abstract group G:

$$\begin{matrix} 1 & s_2 & s_3 & \cdots & s_{g_1} \\ t_2 & s_2t_2 & s_3t_2 & \cdots & s_{g_1}t_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ t_\lambda & s_2t_\lambda & s_3t_\lambda & \cdots & s_{g_1}t_\lambda. \end{matrix} \qquad (A)$$

When the operators of the first row constitute a subgroup G_1 of G, the other rows are known as right co-sets of G with respect to G_1, and when all the operators of (A) are multiplied on the right by any operator of G, the rows of (A) are permuted as units according to a substitution on λ letters, which may be associated with the multiplying operator.

By using successively all the operators of G as multipliers there results a transitive substitution group K of degree λ which is isomorphic with G, and whose order is equal to that of G divided by the order of the largest invariant subgroup of G contained in G_1. In particular, a necessary and sufficient condition that K be simply isomorphic with G is that G_1 does not involve an invariant subgroup of G, besides the identity, and is not itself invariant under G, and a necessary and sufficient condition that K be a primitive substitution group is that G_1 is a maximal subgroup of G.*

The main object of the present article is to give a method for exhibiting fundamental relations between properties of an abstract group G and the subgroups of the isomorphic substitution group K which are separately composed of all the substitutions of K omitting one of its λ letters. Incidentally the following theorem relating to simply transitive primitive substitution groups is proved.

If in a transitive group of degree λ a subgroup composed of all its substitutions which omit a given letter has a transitive constituent of degree $\lambda - \alpha$, $\alpha > 1$, then for a fixed value of α only a finite number of values can be assigned to λ such that the transitive group of degree λ may be primitive.

The method noted in the preceding paragraph depends upon the known

* Cf. W. von Dyck, Mathematische Annalen, vol. 22 (1883), p. 90.

fact that, if we represent the λ rows of (A) by the letters $a_1, a_2, \cdots, a_\lambda$, then these rows are separately composed of all the operators of G which correspond to the substitutions of K which replace a_1 by a given letter. In particular, the first row of (A) is composed of all the operators of G which correspond to the subgroup of K composed of all the substitutions of K which omit a_1.

Suppose that G_1 is a non-invariant subgroup of G and that G_2 is one of the conjugates of G_1. It is known that each of the rows of (A) which involves at least one operator of G_2 must involve the same number of operators of G_2, and that at least one of the rows of (A) contains none of the operators of G_2.* When there is only one such row, it results from the fact noted in the preceding paragraph that the subgroup K_2 of K which corresponds to G_2 replaces one of the λ letters of K by $\lambda - 1$ letters, and hence this subgroup must be transitive and of degree $\lambda - 1$. That is,

A necessary and sufficient condition that K be multiply transitive is that the operators of G_2 appear in $\lambda - 1$ of the rows of (A).

When the operators of G_2 appear in exactly $\lambda - 2$ of the rows of (A), one of these rows besides the first involves operators which are found in each of the conjugates of G_2, since the number of these conjugates is $\lambda/2$. The operators of this row transform G_1 into itself and together with G_1 they constitute a subgroup H_1 of G. When G is represented in the form of a rectangle in which H_1 constitutes the first row, the operators of G_2 will appear in all except one of the rows of (A). A necessary and sufficient condition that H_1 be invariant under G is that $\lambda = 4$. When $\lambda > 4$, a conjugate of H_1 must involve G_2, and hence its operators must appear in all except one of these new rows. As these rows correspond to systems of imprimitivity of K the following theorem has been established:

If the operators of G_2 appear in $\lambda - 2$ of the rows of (A) then K has one and only one set of systems of imprimitivity and transforms these systems according to a multiply transitive group of degree $\lambda/2$.

From what precedes it results that when the operators of G_2 appear either in $\lambda - 1$ or in exactly $\lambda - 2$ of the rows of (A), then the same must be true as regards the operators of all the other conjugates of G_1 except those of G_1 itself. When the operators of G_2 appear in $\lambda - \alpha$ of the rows of (A), $\alpha > 2$, it is not necessarily true that the operators of the other conjugates of G_1, with the exception of G_1, appear in exactly $\lambda - \alpha$ of the rows of (A). This results directly from the fact that K_2 may then have transitive constituents of different degrees. A necessary and sufficient condition that one of these transitive constituents be of degree 2 is that all the operators of a conjugate of G_1 appear in exactly two rows of (A).

* Miller, Blichfeldt, Dickson, Finite Groups 1916, p. 68.

Whenever the operators of G_2 appear in exactly $\lambda - \alpha$, $\lambda > \alpha > 1$, of the rows of (A), the subgroup K_1 composed of all the substitutions of K which omit a given letter has a transitive constituent of degree $\lambda - \alpha$. When K_1 is transitive, then K has only one set of systems of imprimitivity such that each system involved α letters, and it permutes these systems according to a multiply transitive substitution group of degree λ/α, and all the conjugates of G_1, except G_1 itself, have operators in each of the same α rows of (A). When K has other systems of imprimitivity besides those involving α letters, these systems must involve a submultiple of α letters. In particular, when α is a prime number K has only one set of systems of imprimitivity.

When $\alpha > 2$ is a fixed number and K is primitive, K_1 must have a constituent of degree $\alpha - 1$. The subgroup of K_1 which corresponds to the identity of this constituent must be invariant under K_1, and if it is not the identity, it must appear in a conjugate of K_1 without being invariant under this conjugate.* Hence it must have a transitive constituent whose degree cannot exceed $\alpha - 1$. As the order of a substitution group of degree $\alpha - 1$ cannot exceed $(\alpha - 1)!$, the transitive constituent of degree $\lambda - \alpha$ contained in K_1 could not replace one of its letters by more than $(\alpha - 1)\cdot(\alpha - 1)!$ letters. It therefore results that when K is primitive $\lambda - \alpha \leqq (\alpha - 1)\cdot(\alpha - 1)!$ This constitutes a proof of the theorem noted above which may be stated as follows:

If the subgroup composed of all the substitutions of a transitive group of degree λ which omit a given letter has a transitive constituent of degree $\lambda - \alpha$ then this transitive group is imprimitive whenever $\lambda > (\alpha - 1)\cdot(\alpha - 1)! + \alpha$.

In the particular case when $\alpha = 3$ it follows from this theorem that K could not be primitive when $\lambda > 7$. As a matter of fact, it is easy to verify that in this special case K can only be primitive when it is the dihedral group of order 10 and of degree 5. When $\alpha = 4$ there are several possible primitive groups which satisfy the given conditions.

* G. A. Miller, Proceedings of the London Mathematical Society, vol. 28 (1896), p. 534.

THE COMPLETE QUADRILATERAL.

By John Wentworth Clawson.

Introduction. It is the purpose of this paper to discuss systematically the principal points and lines related to the complete quadrilateral, referring, as far as practicable, to the discoverers of theorems which are not new.* In part I more than thirty theorems are given of which (7) and (31) are probably new, the others having been collected from various sources; in part II sixteen new theorems are derived by an inversion; in part III some properties of the cyclic quadrangle are collected, from which, in part IV, by a polar reciprocation, fourteen other new properties of the complete quadrilateral are deduced. The methods of proof used are those of elementary pure geometry.

Throughout the paper ABC will be taken to denote the angle through which the line AB, taken as a whole, must be rotated in order for it to coincide with BC, taken as a whole, the rotation taking place in the positive direction. With this understanding, two angles are regarded as equivalent if they differ only by multiples of π; so $n\pi = 0$. The symbol $=$ is used throughout the paper in this sense.†

As it is impossible to print figures to illustrate all the theorems of this paper, the reader is urged to draw his own figures wherever it is necessary.

Part I. Points and lines related to the complete quadrilateral.

1. **The focal point.** The complete quadrilateral consists of four coplanar straight lines, l_1, l_2, l_3, l_4, intersecting in three pairs of opposite vertices, A_{14}, A_{23}; A_{13}, A_{24}; A_{12}, A_{34}. The lines l_2, l_3, l_4, determine a triangle. Let the center of the circle, $\mathcal{C}_1$, which circumscribes this triangle be C_1. Three other such triangles and circumscribed circles exist.

* In 1828 the famous geometer Steiner proposed for solution (Gergonne's Annales, vol. 18, p. 302) a list of ten theorems relating to the complete quadrilateral. No other notable attempt to collect the properties of this interesting figure appears to have been made until 1901, when Léon Ripert published a paper (Compte Rendu de l'Association francaise pour l'Avancement des Sciences, vol. 30, part 2, p. 91), in which he derived a number of theorems, using the method of trilinear coördinates. Other theorems are scattered through the journals mentioned in the foot notes and elsewhere. Theorems in elementary geometry have appeared in so many places that it is impossible to hope that I have traced all the theorems given to their sources, or that I have not missed altogether some important properties of the quadrilateral.

† R. A. Johnson pointed out in the American Mathematical Monthly, vol. 24 (1917), p. 101, that proofs in elementary geometry may be made to cover all possible figures if the above conventions are adopted.

(1) The four circumscribed circles are concurrent at a point.* This will be called the *focal point* of the quadrilateral, and will be denoted by F.

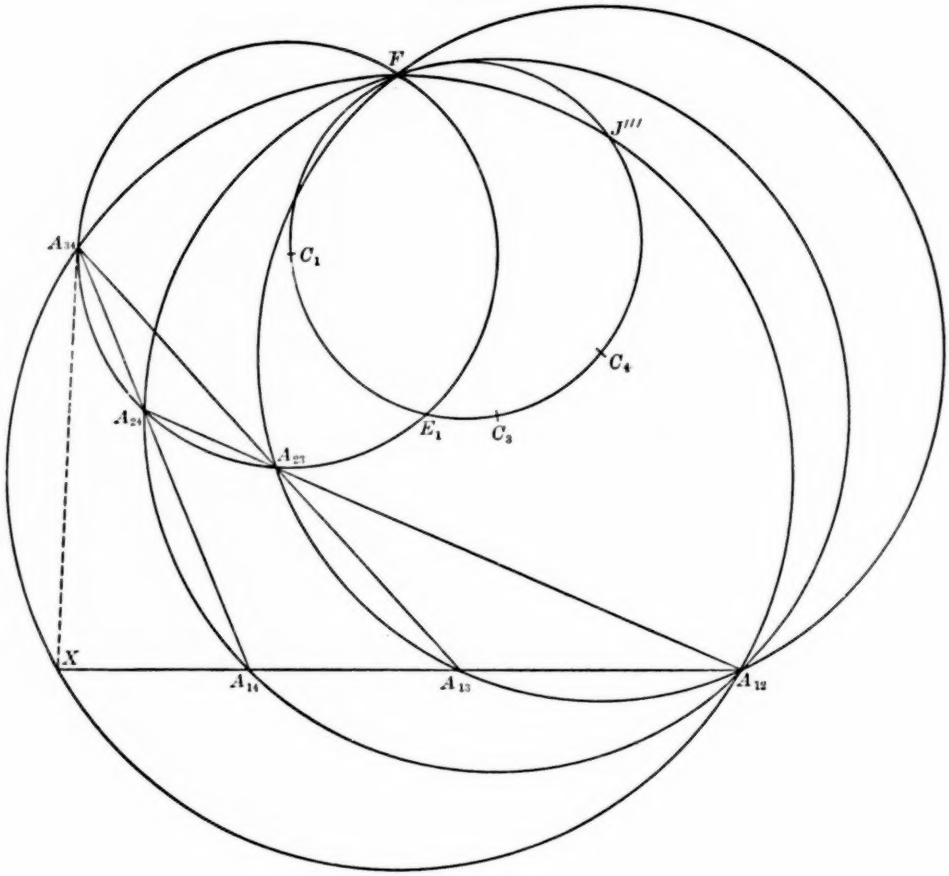

FIG. 1.

For, if F is the intersection of $\mathcal{C}_1$ and $\mathcal{C}_3$ (Fig. 1), considering points on $\mathcal{C}_1$, $A_{24}FA_{23} = A_{24}A_{34}A_{23}$; and, considering points on $\mathcal{C}_3$, $A_{12}FA_{24} = A_{12}A_{14}A_{24}$; by addition of equals, $A_{12}FA_{23} = A_{12}A_{13}A_{23}$; and therefore the circle $\mathcal{C}_4$ also passes through F. Similarly it may be proved that the circle $\mathcal{C}_2$ passes through the focal point.

F has been called the *Wallace point* and the *Miquel point* of the quadri-

* This theorem was first stated by "Scoticus" in 1804 in Leybourn's Mathematical Repository, vol. 1, part 1, p. 170. Mackay, in the Proceedings of the Edinburgh Mathematical Society, vol. 9 (1890), pp. 83–91, says that "Scoticus" is a pseudonym of Dr. William Wallace. It is the first of Steiner's theorems, l. c.; and was proved again by Miquel in Liouville's Journal de Mathématiques, vol. 3 (1838), pp. 485–487. The proof above is that of the Repository modified to suit Johnson's notation.

lateral. It seems wiser, however, to avoid the use of a proper name if a short descriptive name can be coined.

(2) The four points where the tangent at A_{12} to $\mathcal{C}_3$ cuts l_3, where the tangent at A_{12} to $\mathcal{C}_4$ cuts l_4, where the tangent at A_{34} to $\mathcal{C}_1$ cuts l_1, where the tangent at A_{34} to $\mathcal{C}_2$ cuts l_2, are concyclic with A_{12} and A_{34}. This circle and two circles similarly obtained are concurrent at F.*

For, let the tangent at A_{34} to $\mathcal{C}_1$ meet l_1 at X. Then,

$$A_{13}XA_{34} + XA_{34}A_{13} + A_{34}A_{13}X = 0;$$

hence, since $A_{34}X$ is a tangent to $\mathcal{C}_1$, $A_{12}XA_{34} + A_{34}A_{24}A_{23} + A_{34}A_{13}A_{12} = 0$. Again, $A_{12}FA_{34} + FA_{34}A_{24} + A_{34}A_{24}A_{12} + A_{24}A_{12}F = 0$. But, considering points on $\mathcal{C}_2$, $FA_{34}A_{24} = FA_{13}A_{12}$; and, considering points on $\mathcal{C}_4$, $A_{24}A_{12}F = A_{34}A_{13}F$; hence, $A_{12}FA_{34} + FA_{13}A_{12} + A_{34}A_{24}A_{12} + A_{34}A_{13}F = 0$; and therefore $A_{12}FA_{34} + A_{34}A_{24}A_{23} + A_{34}A_{13}A_{12} = 0$. Comparing this with the second equation, we see that $A_{12}XA_{34} = A_{12}FA_{34}$ and hence that A_{12}, X, A_{34}, F are concyclic. Similarly the other facts may be proved.

Draw the three diagonals, n' joining A_{14} to A_{23}, n'' joining A_{13} to A_{24}, n''' joining A_{12} to A_{34}; let n', n'' intersect at D_{12}, this and two points similarly obtained forming the diagonal triangle $D_{12}D_{13}D_{23}$.

Let F_1', F_2', F_1'', F_2'', F_1''', F_2''' be the focal points of the quadrilaterals $n''n'''l_2l_3$, $n''n'''l_1l_4$, $n'n'''l_1l_3$, $n'n'''l_2l_4$, $n'n''l_1l_2$, $n'n''l_3l_4$ respectively. Call the centers of the circles circumscribing l_2l_3n'', l_2l_3n''', C_{23}'', C_{23}''' respectively. Then $C_1C_{23}''C_4C_{23}'''$ is a parallelogram, since C_1C_{23}'' and C_4C_{23}''' are both perpendicular to l_2, and C_1C_{23}''' and C_4C_{23}'' are both perpendicular to l_3. Again, since FA_{23} and $F_1'A_{23}$ are the common chords of circles $\mathcal{C}_1$, $\mathcal{C}_4$ and $\mathcal{C}_{23}''$, $\mathcal{C}_{23}'''$ respectively, the point of intersection of the diagonals of this parallelogram is equally distant from A_{23}, F and F_1'. It is further easily proved that a circle can be drawn with this point as center, passing through A_{23}, F, F_1' and the middle points of $A_{13}A_{34}$ and $A_{12}A_{24}$. In future the notation (AB) will be used for the middle point of the line joining A to B. Hence

(3) The circles determined by A_{23}, $(A_{12}A_{24})$, $(A_{13}A_{34})$; by A_{14}, $(A_{12}A_{13})$, $(A_{34}A_{24})$; and four other such sets of points pass respectively through F_1', F_2', F_1'', F_2'', F_1''', F_2''' and all six circles pass through F.

In exactly the same way it can be proved that

(3a) The circle determined by D_{23}, $(A_{13}A_{24})$ and $(A_{12}A_{34})$ passes through F_1' and F_2'†; and that two other such circles can be drawn.

* This theorem is given as an unproved exercise in Casey's Analytic Geometry, second edition, 1893, Ex. 81, page 535. The proof here given is new.

† The theorem (3a) was set as Question 252 in Journal de Mathématiques élémentaires et spéciales and solved by Rivard in the same Journal, vol. 5 (1881), p. 118. I have expanded this theorem into the group above.

If we anticipate the result of (8), that the middle points of the diagonals are collinear, it is clear that the three circles of (3*a*) are three of the circles circumscribed to triangles of the quadrilateral whose six vertices are D_{12}, D_{13}, D_{23} and the middle points of the three diagonals. Therefore

(3*b*) The circles $F_1'F_2'D_{23}$, $F_1''F_2''D_{13}$ and $F_1'''F_2'''D_{12}$ are concurrent, and the circle $D_{12}D_{13}D_{23}$ passes through the same point.

It may also be proved that

(3*c*) The circles $FF_1'A_{23}$, $FF_2'A_{14}$ and $F_1'F_2'D_{23}$ are concurrent; and that there are two other sets of three concurrent circles, one from the set (3*a*) and two from the set (3).

2. **The circumcentric circle.** We shall prove the theorem:

(4) The points C_1, C_2, C_3, C_4 and F are concyclic.*

The circle, $\mathcal{C}$, on which they lie will be called the *circumcentric circle* of the quadrilateral.

For, $FC_3C_1 = FA_{12}A_{24}$ and $FC_4C_1 = FA_{12}A_{23}$, since a central angle is double an angle at the circumference subtended by the same arc. Therefore F, C_3, C_1 and C_4 are concyclic; similarly it can be shown that F, C_2, C_1 and C_3 are concyclic.

(5) The lines C_2A_{34}, C_3A_{24}, C_4A_{23} are concurrent at a point common to $\mathcal{C}_1$ and $\mathcal{C}$.†

Call this point E_1. Three other points are similarly found.

For, let C_2A_{34} and C_3A_{24} meet at E_1. Then

$$E_1A_{24}A_{34} + A_{24}A_{34}E_1 + A_{34}E_1A_{24} = 0.$$

But

$$E_1A_{24}A_{34} = (\pi/2) - \tfrac{1}{2}A_{14}C_3A_{24} = (\pi/2) - A_{14}A_{12}A_{24};$$

and

$$A_{24}A_{34}E_1 = (\pi/2) - \tfrac{1}{2}A_{34}C_2A_{14} = (\pi/2) - A_{34}A_{13}A_{14}.$$

Adding and substituting in the first equation, we get

$$A_{34}E_1A_{24} = A_{34}A_{13}A_{14} + A_{14}A_{12}A_{24} = A_{34}A_{23}A_{24}.$$

Hence E_1 lies on the circle $\mathcal{C}_1$.

* This is the second of Steiner's theorems, l. c. It was restated by T. S. Davies in Leybourn's Mathematical Repository, vol. 6 (1835), Question 555, and proved there by him. Probably Davies discovered this theorem independently of Steiner; that he had been studying the quadrilateral for some time is evidenced by his statement in connection with the solution of Question 524 in the same volume of the Repository and by J. S. Mackay's remark in the Proceedings of the Edinburgh Mathematical Society, l. c., that Davies had proposed a question in 1821 in the Leeds Correspondent dealing with a property of this figure. After proving the theorem above, Davies proceeds with several other properties including a special case of (5). This circle is called the *center circle* by Gallatly in his Modern Geometry of the Triangle, London, N. D., page 5. It was called the *eight point circle* by Hermes after he found the four new points of (5) on it. The above is essentially Davies' proof.

† This theorem was partially stated and proved by Davies, l. c. The complete statement and proof, substantially equivalent to the above, is due to Hermes, Nouvelles Annales, vol. 18 (1859), page 359.

Again $C_2C_4C_3 = A_{13}FA_{12}$, since C_2C_4 and C_3C_4 are respectively perpendicular to the common chords FA_{13} and FA_{12}; but, considering points on the circle $\mathcal{C}_4$, the latter equals $A_{13}A_{23}A_{12}$ or $A_{34}A_{23}A_{24}$. But $A_{34}A_{23}A_{24}$ was proved above to be equal to $C_2E_1C_3$. Hence E_1 lies on the circle $\mathcal{C}$.

It follows at once that

(5*a*) Tangents at A_{34} to $\mathcal{C}_2$, at A_{23} to $\mathcal{C}_4$, at A_{24} to $\mathcal{C}_3$, meet at a point diametrically opposite to E_1 in $\mathcal{C}_1$;* also that

(5*b*) The pedal line of E_1 with respect to the triangle $l_2l_3l_4$ is parallel to l_1.†

(6) The perpendicular bisectors of $A_{13}A_{14}$ and $A_{23}A_{24}$ intersect at a point on the circumcentric circle.‡

This point will be denoted by T_{34}. There are six such points.

For, $C_1T_{34}C_2 = A_{24}A_{12}A_{14}$, the sides being respectively perpendicular; the latter is equivalent to $A_{24}FA_{14}$, the four points lying on $\mathcal{C}_3$;

$$A_{24}FA_{14} = C_1C_3C_2,$$

since the line of centers of the circles $\mathcal{C}_1$, $\mathcal{C}_3$ is perpendicular to the common chord $A_{24}F$ and C_2C_3 similarly perpendicular to $A_{14}F$. Then

$$C_1T_{34}C_2 = C_1C_3C_2,$$

and the four points are concyclic.

T_{34} is evidently the point diametrically opposite to A_{12} in the fifth of the circles of (3).

The six vertices of the quadrilateral taken in threes determine sixteen circles. Four of these are $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$. Denote by C_{13}', C_{14}'', C_{34}''', $\cdots$ the centers of the circles circumscribing the triangles l_1l_3n', l_1l_4n'', l_3l_4n''', $\cdots$.

It is evident that C_1, C_{24}''' and C_{34}'' are collinear, since the three circles are coaxal. Again C_{12}', C_{13}', C_{24}' and C_{34}' are collinear, for the same reason. It is easily seen then that the sixteen centers lie three by three on twelve lines, the perpendicular bisectors of pairs of vertices on the same l, and four by four on three lines, the perpendicular bisectors of the diagonals.

Now $A_{24}A_{34}C_1 = (\pi/2) - A_{34}A_{23}A_{24} = A_{12}A_{34}C_{23}'''$. Therefore

$$A_{24}A_{34}A_{12} = C_1A_{34}C_{23}''',$$

on adding $C_1A_{34}A_{12}$. And $C_1C_{24}'''C_{23}''' = A_{24}A_{34}A_{12}$, since C_1C_{24}''', $C_{24}'''C_{23}'''$ are perpendicular respectively to l_4, n'''. Therefore

$$C_1C_{24}'''C_{23}''' = C_1A_{34}C_{23}''';$$

* Wetzig, Crelle's Journal, vol. 72 (1870), p. 351.

† Gallatly, Modern Geometry of the Triangle, p. 27.

‡ This theorem is given by Ripert, l. c., who proves it analytically. I have furnished the above proof.

and C_1, C_{23}''', C_{24}''' and A_{34} are concyclic. Similarly C_2, C_{13}''', C_{14}''', A_{34}; C_3, C_{14}''', C_{24}''', A_{12}; C_4, C_{13}''', C_{23}''', A_{12} are concyclic.

Let the circles $C_1C_{24}'''A_{34}$ and $C_2C_{13}'''A_{34}$ intersect again at J'''. Now $C_1J'''A_{34} = C_1C_{24}'''A_{34} = A_{24}A_{12}A_{34}$; and

$$C_2J'''A_{34} = C_2C_{13}'''A_{34} = A_{13}A_{12}A_{34}.$$

Therefore $C_1J'''C_2 = A_{23}A_{12}A_{13} = A_{24}A_{12}A_{14} = C_1C_3C_2$, by (6). Consequently J''' is on the circumcentric circle. Similarly we prove that

(7) The four circles determined by the circumcenters of (*a*) the three triangles determined by A_{34} and two of the vertices on l_1, (*b*) the three triangles determined by A_{34} and two of the vertices on l_2, (*c*) the three triangles determined by A_{12} and two of the vertices on l_3, (*d*) the three triangles determined by A_{12} and two of the vertices on l_4 are concurrent at J''', a point on the circumcentric circle.*

Two other points are associated with the other pairs of opposite vertices.

Moreover, since $C_1J'''A_{34} = C_1C_{24}'''A_{34} = A_{24}A_{12}A_{34}$; and

$$C_1J'''F = C_1C_3F = A_{24}A_{12}F;$$

therefore $A_{34}J'''F = A_{34}A_{12}F$. Hence

(7*a*) The points A_{34}, A_{12}, F and J''' are concyclic; similarly A_{13}, A_{24}, F, J'' and A_{14}, A_{23}, F, J' are concyclic.*

These are the circles of (2).

3. **The mid-diagonal line.** We shall prove that

(8) The middle points of the diagonals of the quadrilateral are collinear on a line, m, which contains the centers of all conics touching the four sides of the quadrilateral.†

This line will be called the *mid-diagonal line* of the quadrilateral; Ripert calls it the *axis of mean distances;* it is sometimes called the *Newtonian.*

Let one of the inscribed conics touch l_1 at Z_1, l_4 at Z_4, l_3 at Z_3. Then, applying Brianchon's theorem to the hexagons $Z_4A_{24}A_{12}Z_1A_{13}A_{34}$, $Z_3A_{23}A_{12}Z_1A_{14}A_{34}$, we see that Z_4Z_1, Z_3Z_1 pass respectively through D_{23}, D_{13}. Let the lines joining A_{14}, A_{13} respectively to (Z_4Z_1), (Z_3Z_1) meet at X. Then X is the center of the inscribed conic.

Draw $A_{14}Y$, $A_{13}Y$ parallel respectively to Z_4Z_1, Z_3Z_1 intersecting at Y.

* Theorems (7) and (7*a*) are new.

† This theorem is ascribed to Newton by Steiner, l. c., and others; it is sometimes ascribed to Gauss, for example by Schlömilch, Berichte der Gesellschaft der Wissenshaften zu Leipzig, vol. 9, 1854, p. 4, and by Hall, Messenger of Mathematics, vol. 4, 1868, p. 137. In fact the theorem is an immediate consequence of lemma XXV, cor. 3, sect. V, vol. 1 of the Principia, 1684. The first part of it was probably first explicitly stated by Connor in the Ladies' Diary for 1795. The proof above is due to Poncelet, Gergonne's Annales, vol. 12, p. 109. The mid-diagonal is thus the earliest known of the lines and points connected with the complete quadrilateral.

Then the pencils of rays with vertices at A_{14} and A_{13} respectively and passing through the points A_{34}, Y, Z_1, X are both harmonic. Hence A_{34}, Y, X are collinear.

Draw YT parallel to n''', cutting l_1 at T. Now triangles $TA_{13}Y$, $A_{12}Z_1D_{13}$ are similar, so that $TA_{13} \cdot D_{13}A_{12} = TY \cdot Z_1A_{12}$; and triangles $TA_{14}Y$, $A_{12}Z_1D_{23}$ are similar, so that $TA_{14} \cdot D_{23}A_{12} = TY \cdot Z_1A_{12}$. Hence $TA_{13}/TA_{14} = D_{23}A_{12}/D_{13}A_{12}$; and therefore $TA_{13}/A_{13}A_{14} = D_{23}A_{12}/D_{23}D_{13}$. Hence T is a fixed point; and therefore, if we consider the various possible inscribed conics, the locus of Y is a fixed line YT.

Now the pencil having T as vertex formed by rays passing through A_{34}, Y, Z_1, X is harmonic; therefore the locus of X is the fixed line XT. Let XT cut $A_{12}A_{34}$ in B_3. Then the points A_{34}, ∞, A_{12}, B_3 form a harmonic range; that is B_3 bisects $A_{12}A_{34}$.

Similarly we can show that the line which is the locus of centers of the inscribed conics bisects $A_{13}A_{24}$ at B_2 and $A_{14}A_{23}$ at B_1.*

Let U_1 be the center of gravity of the triangle $l_2l_3l_4$, U'_1 the center of gravity of equal masses placed at the three vertices that lie on the line l_1. Then

(9) The line U_1U_1' is concurrent with the three similar lines at their common mid-point U on the mid-diagonal line.†

The point U is the center of gravity of equal masses placed at (*a*) the six vertices; (*b*) the four centroids U_1, U_2, U_3, U_4; (*c*) the four points U_1', U_2', U_3', U_4'; (*d*) the three points B_1, B_2, B_3.

Place equal masses, m, at the six vertices. (*a*) Combine masses at opposite vertices; this gives three masses, $2m$, at B_1, B_2, B_3 whose center of gravity, U, is found by bisecting B_2B_3 at C and locating CU equal to one third of CB_1. (*b*) Replace three masses, m, at A_{34}, A_{24}, A_{23} by $3m$ at U_1, replace three masses, m, at A_{12}, A_{13}, A_{14} by $3m$ at U'. Then U bisects U_1U_1', and it is also the common mid-point of the three lines joining the mid-points of the opposite pairs of sides of the quadrangle $U_1U_2U_3U_4$.

(10) The lines joining $(A_{13}A_{14})$ to $(A_{23}A_{24})$ and $(A_{13}A_{23})$ to $(A_{14}A_{24})$ intersect on m.‡

Similarly two other points on m are located. The theorem follows at once from the fact that the diagonals of a parallelogram bisect each other.

4. The pedal line.

* Simpler proofs that the middle points of the diagonals are collinear may be found in Casey's Sequel to Euclid, 1886, p. 5, in C. V. Durell's Plane Geometry for Advanced Students, Part I, 1909, pp. 118, 119, 188, and elsewhere.

Some remarkable members of the family of conics touching the four sides of the quadrilateral are discussed by Ripert, l. c.

† This theorem is given by Ripert, l. c.; it is proved, as above, by Neuberg, Annales de la Société scientifique de Bruxelles, 1902, p. 13.

‡ This theorem is given by Schlömilch, l. c.

(11) The feet of the perpendiculars from the focal point to the four sides of the quadrilateral are collinear.*

The line on which these four points lie will be referred to as the *pedal line* of the quadrilateral and it will be denoted by p.

Let G_1 be the foot of the perpendicular from F to l_1. The theorem follows at once from the Simson, Wallace or pedal line theorem.†

(12) At G_1 the circles drawn on FA_{12}, FA_{13}, FA_{14} as diameters concur.‡

Three circles similarly concur at G_2, G_3, G_4 on the pedal line.

If A, B, C, D are four concyclic points and H_a, H_b, H_c, H_d are the orthocenters of the triangles formed by omitting A, B, C, D respectively, it is easily seen that AH_a, BH_b, CH_c, DH_d bisect each other at a point N. So H_a, H_b, H_c, H_d is a group of four concyclic points on a circle whose radius is equal to that of the circle circumscribing $ABCD$.§ From the fact that the orthocenter of a triangle is the external center of similitude of the nine-point circle and the circumcircle of the triangle, and that the radii of these circles are in the ratio of 1 : 2, it follows that the nine-point circles of the four triangles ABC, ABD, ACD, BCD are concurrent at N. From the fact that the line joining a point on the circumcircle of a triangle to the orthocenter of the triangle is bisected by the pedal line of the first-named point, it follows that the pedal lines of D, A, B, C with respect to the triangles ABC, BCD, CDA, DAB respectively all pass through the point N.‖ Therefore, returning to the complete quadrilateral, we have

(13) The nine-point circle of the triangle $A_{23}A_{34}A_{24}$ is cut by the nine-point circles of $FA_{23}A_{34}$, $FA_{34}A_{24}$, $FA_{23}A_{24}$ at the point on the pedal line where FH_1 cuts it, H_1 being the orthocenter of the triangle $A_{23}A_{34}A_{24}$.

Three similar points exist.

Let F_1 be the point in $\mathcal{C}_1$ diametrically opposite to F. Draw F_1G_2', F_1G_3' perpendicular respectively to l_2, l_3; then $G_2'G_3'$ is the pedal line of F_1 with respect to the triangle $l_2l_3l_4$.

Now $G_3G_2A_{23} = G_3FA_{23} = (\pi/2) - FA_{23}A_{34}$; and

$$G_3'G_2'A_{23} = G_3'F_1A_{23} = (\pi/2) - A_{34}A_{23}F_1.$$

Therefore $FA_{23}A_{34} + A_{34}A_{23}F_1 = (\pi/2) = G_3'G_2'A_{23} + G_3G_2A_{23}$. Hence the pedal line of F_1 is perpendicular to the pedal line of the quadrilateral. Similarly the pedal lines of F_2, F_3, F_4 with respect to the triangles $l_3l_4l_1$, $l_4l_1l_2$, $l_1l_2l_3$ are perpendicular to the pedal line of the quadrilateral.

* This theorem was given by Steiner, l. c., and proved by Davies, l. c.

† Mackay, l. c.; Mathematical Repository, old series, vol. 2, 1800, p. 111.

‡ Catalan, Théorèmes et problèmes, sixth edition, 1865, p. 34.

§ Mention, Nouvelles Annales, vol. 4 (1845), p. 654. Cf. (ϵ) and (μ) in part III.

‖ These facts are quoted by Alison, Proceedings of the Edinburgh Mathematical Society, vol. 3 (1885), pp. 79–93; they are referred by him to the Ladies' and Gentlemen's Diary for 1864, p. 55, and to the Reprints from the Educational Times, vol. 1, Question 1431.

Now let H_1F meet G_2G_3 at X_1, H_1F_1 meet $G_2'G_3'$ at X_1'. Let Y_1 be the middle point of C_1H_1. Then, since X_1 bisects FH_1, X_1' bisects F_1H_1, Y_1 bisects C_1H_1 and since C_1 is the center of the circle whose diameter is FF_1, Y_1 is the center of the circle whose diameter is X_1X_1'. But Y_1 is the center of the nine-point circle of the triangle $l_2l_3l_4$, and X_1, X_1' are on this circle. Hence G_2G_3 and $G_2'G_3'$ intersect at a point on the nine-point circle of $l_2l_3l_4$. Thus the pedal lines of the extremities of any diameter of the circumcircle of a triangle intersect at right angles on the nine-point circle of the triangle.* Hence

(14) The nine-point circle of the triangle $A_{23}A_{34}A_{24}$ is cut by the pedal line of F_1 with respect to that triangle at a second point on the quadrilateral's pedal line.†

Three similar points exist.

The following generalization of the pedal line may be mentioned:

(11*a*) If FG_1, FG_2, FG_3, FG_4 are drawn making any constant angle with l_1, l_2, l_3, l_4, the points G_1, G_2, G_3, G_4 are collinear.‡

3. **The orthocentric line.** From the well-known fact that the pedal line of a point with respect to a triangle bisects the join of that point to the orthocenter of the triangle,§ it follows that

(15) The orthocenters of the four triangles formed by the sides of the quadrilateral in threes are collinear in a line parallel to the pedal line and twice as far from the focal point as is the pedal line.‖

This line will be called the *orthocentric line*; it will be denoted by o.

(16) Circles drawn on the three diagonals of the quadrilateral as diameters are coaxal, their radical axis being the orthocentric line.¶

Let $A_{34}H_1$ cut l_2 at A_2, $A_{24}H_1$ cut l_3 at A_3, $A_{23}H_1$ cut l_4 at A_4. By similar triangles, $A_{34}H_1 \cdot H_1A_2 = A_{24}H_1 \cdot H_1A_3 = A_{23}H_1 \cdot H_1A_4$. But $A_{34}H_1 \cdot H_1A_2$ is equal to the square of the tangent from H_1 to the circle on $A_{34}A_{12}$ as diameter; and $A_{24}H_1 \cdot H_1A_3$ is equal to the square of the tangent from H_1 to the circle on $A_{24}A_{13}$ as diameter. These tangents may be imaginary. Then H_1 is a point on the radical axes of the three circles in pairs. Similarly H_2, H_3, H_4 are on these three radical axes. But, by (15), it follows that the three radical axes coincide with the orthocentric line.

As an immediate consequence we have the theorem

* Casey, Sequel to Euclid, edition of 1886, Ex. 138, p. 164. Attributed to Graham in this, not in later, editions.

† In (13) and (14) I have made an obvious extension of theorems in the geometry of the triangle to the geometry of the quadrilateral.

‡ Poncelet, Propriétés Projectives, 1822, § 468.

§ E. g., Casey, Sequel to Euclid, p. 36.

‖ This theorem is given by Steiner, l. c., and is proved by Davies, l. c.

¶ This has been called Bodenmiller's theorem, for example by Schlömilch, l. c. It was stated without proof by Davies, l. c., in 1835; it is not in Steiner's list of 1828.

(16*a*) The mid-diagonal line is perpendicular to the orthocentric and pedal lines.*

Also

(17) The parabola π which touches the sides of the quadrilateral has F for its focus and the orthocentric line for its directrix.†

For the circle circumscribing the triangle formed by any three of these tangents passes through the focus; and the directrix passes through the orthocenter of that triangle.‡

(17*a*) The pedal line and the family of lines of (11*a*) all touch the parabola π.†

(17*b*) The parabola π also touches the six lines joining $(A_{13}A_{14})$ to $(A_{23}A_{24})$; $(A_{13}A_{23})$ to $(A_{14}A_{24})$; $\cdots$.§

This is easily proved by the method of polar reciprocation, as in part IV below, since the harmonically conjugate lines of L_1F with respect to the angle $L_3L_1L_4$ and of L_2F with respect to the angle $L_3L_2L_4$ are easily seen to concur at a point on the circumcircle, which is the reciprocal of the parabola.

(17*c*) The ends of chords parallel to l_1 from A_{34} in $\mathcal{C}_2$, from A_{24} in $\mathcal{C}_3$, from A_{23} in $\mathcal{C}_4$ are collinear with F in a line which cuts π at the point of tangency of the parabola with l_1.‖

Three other such lines exist.

For, if the line from A_{34} parallel to l_1, cuts $\mathcal{C}_2$ at U_2, join FU_2 and let it cut l_1 at T. Now, considering points on $\mathcal{C}_2$, $FU_2A_{34} = FA_{13}A_{34}$; and, considering that F, A_{13}, G_1, G_3 are concyclic, the latter is equal to FG_1G_3. Let the axis of π be drawn perpendicular to G_1G_3 and let it cut l_1 at S. Then $TSF = FG_1G_3 = FU_2A_{34}$. But, by alternate-interior angles, the latter is equal to FTS. Therefore, since $TSF = FTS$, T must be the point of contact of l_1 with the parabola π.

Similarly FU_3 and FU_4 cut l_1 at this point.

Draw $A_{24}L_3'$, $A_{34}L_2'$ perpendicular to l_1; draw $L_3'M_3'$, $L_2'M_2'$ perpendicular respectively to l_3, l_2. Let these lines intersect at V_1.

Now $A_{24}L_3'$ passes through H_3, $A_{34}L_2'$ through H_2. Let $L_3'V_1$, $A_{34}H_2$ cut at Y; $L_2'H_2$, $A_{14}H_3$ at Z. Then the anharmonic ratio of the pencil whose vertex is L_3' and whose rays pass through H_3, V_1, H_2, A_{12} is equal to the anharmonic ratio¶ of its transversal $(\infty\, YH_2L_2')$; but this is equal to H_2L_2'/H_2Y, which is equal to $A_{14}L_2'/A_{14}L_3'$.

* This theorem is given by Steiner, l. c.

† This theorem is generally credited to Steiner, for example by S. Kantor, Sitzungsberichte der Akademie der Wissenschaften, Wien, vol. 76, part 2 (1878), p. 753. It is not however given in Steiner's note of 1828.

‡ Salmon, Conic Sections, edition of 1879, pp. 207, 275. C. Smith, Geometrical Conics, edition of 1899, pp. 48, 70.

§ This theorem is given by Ripert, l. c., and proved by him analytically.

‖ This theorem is new.

¶ The anharmonic ratio of four points on a line $(ABCD)$ is the value of $(AB\cdot CD/AD\cdot CB)$.

Again the anharmonic ratio of the pencil whose vertex is L_2' and whose rays pass through H_3, V_1, H_2, A_{12} is equal to the anharmonic ratio of its transversal ($H_3 \propto ZA_{14}$); but this is equal to $ZA_{14}/A_{14}H_3$, which is equal to $A_{14}L_2'/A_{14}L_3'$.

Therefore the pencils have equal anharmonic ratios and H_3, V_1, H_2 are collinear.

Similarly, if $A_{23}L_4'$ is perpendicular to l_1 and L_4M_4' is perpendicular to l_4, this last line passes through V_1.* V_1 is called the *orthopole* of l_1 with respect to the triangle $l_2l_3l_4$. Hence

(18) If perpendiculars are drawn from the vertices of one of the four triangles of the quadrilateral to the other side of the quadrilateral, and if then from the feet of these perpendiculars are drawn to the sides of the original triangle opposite the respective vertices, these three lines are concurrent at a point on the orthocentric line.

There are four such points.

(19) The center of the circle, $\mathfrak{D}$, circumscribing the diagonal triangle $D_{12}D_{13}D_{23}$ lies on the orthocentric line.†

For $(A_{13}D_{12}A_{24}D_{23})$ is a harmonic range; therefore any circle through D_{23} and D_{12} is orthogonal to the circle on $A_{13}A_{24}$ as diameter. Consequently the circle $\mathfrak{D}$ is orthogonal to the circles having $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ respectively as diameters. Hence, the theorem follows from (16).

Let $A_{23}H_1$ cut $A_{14}H_2$ in H_{12}' and $A_{14}H_3$ in H_{13}'; $A_{23}H_4$ cut $A_{14}H_2$ in H_{24}' and $A_{14}H_3$ in H_{34}'. Let $A_{23}H_1$ cut l_1 at S; let the orthocentric line cut l_1 at T_1. Now, since lines joining corresponding points are perpendicular to l_4, the anharmonic ratios of $(ST_1A_{13}A_{12})$ and $(H_1T_1H_2H_3)$ are equal; considering them as transversals across a pencil with vertex at A_{14}, we see that the anharmonic ratios of $(H_1T_1H_2H_3)$ and $(H_1SH_{12}'H_{13}')$ are equal; but the latter is the same as $(SH_1H_{13}'H_{12}')$. Hence

$$(ST_1A_{13}A_{12}) = (SH_1H_{13}'H_{12}');$$

and $A_{13}H_{13}'$, $A_{12}H_{12}'$ and the orthocentric line are concurrent. Thus it can be shown that

(20) The lines $A_{12}H_{12}'$, $A_{13}H_{13}'$, $A_{14}H_{14}'$, $A_{34}H_{34}'$ are concurrent at a point Z', on the orthocentric line.‡

Two other points are similarly found.

Let $A_{23}H_1$ cut l_4 at L_1^{iv} and $A_{23}H_4$ cut l_1 at L_4'. It is not hard to show that

* According to Gallatly, this theorem is due to Neuberg. See his Modern Geometry of the Triangle, Chapter 6, where this and a number of other properties connected with the orthopole are worked out analytically. I have supplied the above proof and extended the theorem to the quadrilateral.

† Möbius, Berichte der Gesellschaft der Wissenschaften zu Leipzig, vol. 9 (1854), p. 87.

‡ This theorem is due to Terrier, Nouvelles Annales de Mathématiques, vol. 14 (1875), p. 514. I have supplied the proof.

(20*a*) The lines $L_1^{iv}L_4'$ and $L_2'''L_3''$ are concurrent at the point Z' of (20).*

Similarly a pair of lines passes through Z'' and a pair through Z'''.

It is obvious that

(21) The perpendiculars from $(A_{13}A_{14})$ to l_2 and from $(A_{23}A_{24})$ to l_1 are concurrent at (H_3H_4).*

There are six points of this kind on the orthocentric line.

(22) If lines are drawn from each vertex of the quadrilateral isogonally conjugate, with respect to the angle at that vertex, to the line joining that vertex to the focal point of the quadrilateral, these six lines are all perpendicular to the orthocentric line.†

For, F, A_{24}, G_2, G_4 are concyclic; hence, $FG_2G_4 = FA_{24}A_{34}$. Draw $A_{24}T$ isogonally conjugate to‡ $A_{24}F$ with respect to $A_{12}A_{24}A_{34}$. Then $FG_2G_4 = A_{23}A_{24}T$. But FG_2 is perpendicular to l_2; therefore $A_{24}T$ is perpendicular to p; that is, $A_{24}T$ is perpendicular to o.

(23) Perpendiculars from F to l_3, l_1 cut $\mathcal{C}_1$, $\mathcal{C}_3$ respectively in points collinear with A_{24} in a line parallel to o.§

Five other such points exist.

For, $FG_1G_4 = FA_{14}A_{24} = FA_{12}A_{24} = FW_3A_{24}$, where W_3 is the point where FG_1 cuts $\mathcal{C}_3$, the first equation following from the fact that F, G_1, A_{14}, G_4 are concyclic. Therefore W_3A_{24} is parallel to p; similarly W_1A_{24} is parallel to p.

In the quadrilateral $A_{13}A_{14}A_{24}A_{23}$ denote the lengths of the sides in the order named by s_1, s_4, s_2, s_3. Let the perpendicular bisectors of s_1 and s_2 meet at T_{34}, and the perpendicular bisectors of s_3 and s_4 at T_{12}. Compare (6). Denote the distance of T_{12} from s_3 by t_{12}'''.

Now

$$T_{34}B_2^2 - T_{12}B_2^2 = \frac{T_{34}A_{24}^2 + T_{34}A_{13}^2 - T_{12}A_{24}^2 - T_{12}A_{13}^2}{2}$$

$$= \frac{s_2^2 + s_1^2 - s_3^2 - s_4^2}{8} + \frac{t_{34}''^2 + t_{34}'^2 - t_{12}^{iv2} - t_{12}'''^2}{2}.$$

Also

$$T_{34}B_1^2 - T_{12}B_1^2 = \frac{T_{34}A_{23}^2 + T_{34}A_{14}^2 - T_{12}A_{23}^2 - T_{12}A_{14}^2}{2}$$

$$= \frac{s_2^2 + s_1^2 - s_3^2 - s_4^2}{8} + \frac{t_{34}''^2 + t_{34}'^2 - t_{12}^{iv2} - t_{12}'''^2}{2}.$$

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* Terrier, l. c.

† Lachlan, Modern Pure Geometry, 1893, ex. 4, p. 68.

‡ Two lines are *isogonally conjugate* with respect to an angle, if they pass through its vertex and make equal but oppositely directed angles with the sides of the angle.

§ This is an extension of a theorem in Richardson and Ramsay, Modern Pure Geometry, 1893, ex. 15, p. 53.

Hence $T_{34}B_1^2 - T_{12}B_1^2 = T_{34}B_2^2 - T_{12}B_2^2$. Therefore $T_{12}T_{34}$ is perpendicular to B_1B_2. Hence

(24) The chords $T_{12}T_{34}$, $T_{13}T_{24}$, $T_{14}T_{23}$ are all parallel to the orthocentric line.*

Now T_{34} is diametrically opposite to A_{12} in the circle determined by $(A_{13}A_{14})$, $(A_{23}A_{24})$, A_{12}. But this circle goes through F, by (3). Hence FT_{34} is perpendicular to FA_{12}. Hence

(24a) If A_{12}''' is the point diametrically opposite to A_{12} in the circle $\mathfrak{C}_3$, F, T_{34}, A_{12}''', A_{12}^{iv} are collinear.*

Five other sets of collinear points exist, all six lines being concurrent at F.

If T_{34}' is the point in the circumcentric circle diametrically opposite to T_{34}, then $T_{34}FT_{34}' = T_{34}FA_{12} = \pi/2$. Hence

(24b) The line $A_{12}F$ cuts the circumcentric circle at a point diametrically opposite to T_{34}.*

The following theorem is added, without proof:

(24c) The points F, A_{34}'', A_{24}''', A_{23}^{iv} are concyclic; and the center of this circle is diametrically opposite to C_1 on the circumcentric circle.*

There are three other such points.

The following interesting but isolated theorem may be added here without proof:

(25) The perpendicular bisectors of C_1H_1, C_2H_2, C_3H_3 and C_4H_4 are concurrent.†

6. **The bisectors of the angles.** Let the bisector of the angle through which l_1 must be rotated in the positive direction in order for it to coincide with l_2 be denoted by l_{12}. Let l_{14} and l_{23} intersect at K_{11}'; l_{14} and l_{32} at K_{12}'; l_{41} and l_{23} at K_{21}'; l_{41} and l_{32} at K_{22}'. l_{13} and l_{24} at K_{11}''; l_{13} and l_{42} at K_{12}''; l_{31} and l_{24} at K_{21}''; l_{31} and l_{42} at K_{22}''; l_{12} and l_{34} at K_{11}'''; l_{12} and l_{43} at K_{12}'''; l_{21} and l_{34} at K_{21}'''; l_{21} and l_{43} at K_{22}'''. Let the incenter of $l_2l_3l_4$ be I_1; the excenters opposite A_{34}, A_{24}, A_{23} respectively I_{12}, I_{13}, I_{14}. All these points are shown in Fig. 2.

(26) The lines

$$K_{11}'K_{11}''K_{22}''',\ K_{11}'K_{22}''K_{11}''',\ K_{22}'K_{11}''K_{11}''',\ K_{22}'K_{22}''K_{22}'''$$

are the sides of a complete quadrilateral, $\mathfrak{C}_1$, and

$$K_{12}'K_{12}''K_{12}''',\ K_{12}'K_{21}''K_{21}''',\ K_{21}'K_{12}''K_{21}''',\ K_{21}'K_{21}''K_{12}'''$$

are the sides of a complete quadrilateral, $\mathfrak{C}_2$.‡

* This is given and proved analytically by Ripert, l. c.

† Lachlan, Modern Pure Geometry, 1893, Ex. 4, p. 92. Lachlan credits this theorem to Hervey, Educational Times Reprint, vol. 54. Another proof is given by Liénard in Mathesis, vol. 1 (1911), pp. 89–91.

‡ Mention, Nouvelles Annales, vol. 1 (1862), p. 76.

For the lines joining corresponding vertices of the triangles $I_{14}I_{13}I_{12}$, $I_{41}I_{32}I_2$ concur at I_1; hence, by Desargues' Theorem, the intersections of corresponding sides, i. e., K_{11}', K_{22}'', K_{11}''', are collinear. Similarly for the other sets of points.

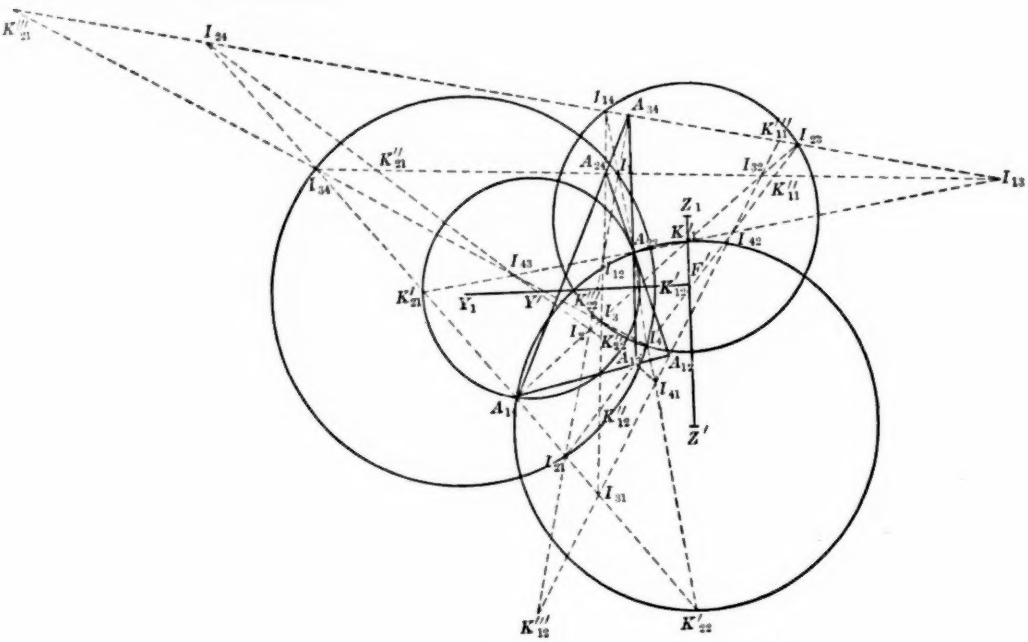

Fig. 2.

Now I_1, I_{21}, I_{34}, I_4 are concyclic. For

$$I_4I_{34}I_1 = \frac{A_{12}A_{24}A_{34} - A_{24}A_{12}A_{14}}{2};$$

and

$$I_4I_{21}I_1 = \frac{A_{12}A_{13}A_{34} - A_{14}A_{34}A_{13}}{2};$$

but $A_{12}A_{24}A_{34} + A_{14}A_{34}A_{13} = A_{24}A_{12}A_{14} + A_{12}A_{13}A_{34}$.

The diagonal triangle of the inscribed quadrangle $I_1I_4I_{21}I_{34}$ is $K_{22}'K_{11}''K_{22}'''$. Now the orthocenter of the diagonal triangle of an inscribed quadrangle is at the center of the circle.* But $K_{22}'K_{11}''K_{22}'''$ is one of the triangles of $\mathfrak{C}_1$. In this way it is proved that

(27) The sixteen centers of the circles inscribed and escribed to the four triangles of the quadrilateral are four by four concyclic, giving rise to eight new circles. These eight circles divide themselves into two groups,

* Steiner, Crelle's Journal, vol. 30. Kantor, l. c. For $N'N''N'''$ is a self-conjugate triangle with respect to the circle.

such that the centers of four circles of one group are collinear (on s_1 the orthocentric line of the quadrilateral $\mathfrak{C}_1$) and those of the other group are also collinear (on s_2 the orthocentric line of the quadrilateral $\mathfrak{C}_2$).*

The circles are I_1, I_{21}, I_{34}, I_4, center Y_1; I_{12}, I_2, I_3, I_{43}, center Y_2; $I_{13}, I_{23}, I_{32}, I_{42}$, center Y_3; $I_{14}, I_{24}, I_{31}, I_{41}$, center Y_4. In the other group, I_{14}, I_{23}, I_3, I_4, center Z_1; $I_{13}, I_{24}, I_{34}, I_{43}$, center Z_2; $I_{12}, I_{21}, I_{31}, I_{42}$, center Z_3; I_1, I_2, I_{32}, I_{41}, center Z_4.

Now

$$Y_1I_1I_4 + I_4I_{14}Z_1 = (\pi/2) - I_4I_{21}I_1 + \pi/2 - I_{14}I_{23}I_4$$
$$= \pi - I_{23}I_{21}A_{34} - A_{34}I_{23}I_{21} = \pi/2.$$

Therefore $Y_1I_4Z_1 = \pi/2$. Hence z_1 is orthogonal to y_1. Hence

(27*a*) The circles of the first group, y, are respectively orthogonal to those of the second group, z; hence the lines s_1, s_2 are perpendicular.†

These lines will be called the *incentric lines* of the quadrilateral.

Now K_{22}' is the pole of $K_{11}''K_{22}'''$ with respect to y_1; therefore, since $K_{11}', K_{11}'', K_{22}'''$ are collinear, K_{22}' and K_{11}' are conjugate points with respect to y_1. Hence the circle on $K_{22}'K_{11}'$ as diameter is orthogonal to y_1. Similarly circles on $K_{11}''K_{22}''$ and $K_{11}'''K_{22}'''$ as diameters are orthogonal to y_1, and these three circles are also orthogonal to y_2, y_3, y_4. In this way we find that

(27*b*) Three new circles may be added to each of the systems, y and z, of mutually orthogonal circles.‡

These circles are $K_{12}'A_{14}A_{23}K_{21}'$, center Y'; $K_{12}'', A_{13}, A_{24}, K_{21}''$, center Y''; $K_{12}''', A_{12}, A_{34}, K_{21}'''$, center Y'''. In the other group, $K_{11}', A_{14}, A_{23}, K_{22}'$, center Z'; $K_{11}'', A_{13}, A_{24}, K_{22}''$, center Z''; $K_{11}''', A_{12}, A_{34}, K_{22}''$, center Z'''.

It follows that

(28) The orthocentric line of the quadrilateral $\mathfrak{C}_1$ is the mid-diagonal line of $\mathfrak{C}_2$, and vice versa.†

Now K_{12}' is the orthocenter of the triangle $K_{11}'K_{21}'K_{22}'$. The nine-point circle of this triangle passes through A_{14} and A_{23}. It has the line joining $(K_{12}'K_{21}')$ to $(K_{11}'K_{22}')$ as diameter. But these are points, Y', Z', on s_1, s_2 respectively, which are perpendicular lines. Hence

(29) The intersection of s_1 and s_2 is on the nine-point circle of $K_{11}'K_{21}'K_{22}'$.†

Call this intersection F'. Then, considering points on the nine-point

* With the exception of the words in parentheses, this theorem is given by Steiner, l. c. The words in parentheses were added by Mention, l. c., who first proved the theorem. The above proof follows Mention closely, with a changed notation.

† Mention, l. c.

‡ Sancery, Nouvelles Annales, vol. 14 (1875), p. 145.

circle of $K_{11}'K_{21}'K_{22}'$, we have $A_{14}F'A_{23} = A_{14}Z'A_{23}$; but, since Z' is the center of the circle $A_{14}K_{11}'A_{23}$, the latter is equal to $2A_{14}K_{22}'A_{23}$ or $2(K_{22}'A_{14}A_{24} + A_{14}A_{24}A_{23} + A_{24}A_{23}K_{22}')$ or $A_{12}A_{14}A_{24} + A_{24}A_{23}A_{34} + 2A_{14}A_{24}A_{23}$ or $A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}$.

But, F being the focal point of the quadrilateral,

$$A_{14}FA_{23} = A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}.*$$

For, $A_{13}FA_{23} = A_{13}A_{12}A_{23}$ and $A_{14}FA_{13} = A_{14}A_{34}A_{13}$.

Hence F' lies on the circle $A_{14}A_{23}F$; similarly on $A_{13}A_{24}F$ and on $A_{12}A_{34}F$. Since these three circles cannot be coaxal,† they have only the point F in common, so F' coincides with F. Hence

(30) The incentric lines intersect at the focal point of the quadrilateral.‡

The interesting fact seems not to have been noticed that

(31) The incentric lines bisect the angles $A_{14}FA_{23}$, $A_{13}FA_{24}$, $A_{12}FA_{34}$.

For, if $K_{12}'K_{21}'$ cuts $K_{11}'K_{22}'$ at W, then A_{14}, Y', A_{23}, F, W and Z' are all points on the nine-point circle of $K_{11}'K_{21}'K_{22}'$. Then

$$A_{14}FY' = A_{14}WY' = A_{14}WK_{21}'.$$

Also $Y'FA_{23} = Y'WA_{23} = K_{21}'WA_{23}$. But $A_{14}W$ and $A_{23}W$ are equally inclined to the altitude $K_{21}'W$. Therefore $A_{14}FY' = Y'FA_{23}$.

Part II. Relations derived by an inversion.

7. Inversion of a quadrilateral. It is well known that a complete quadrilateral, $\mathcal{C}$, inverts with respect to its focal point as center of inversion into the four circumcircles of a second quadrilateral.§ This new quadrilateral, $\mathcal{C}'$, is inversely similar to the original one,‖ the axis of similitude being one of the incentric lines; and points and lines connected with the new figure invert into points and lines connected in different ways with the original one. The incentric lines of the old are evidently also the incentric lines of the new quadrilateral. If we denote by $\mathcal{C}_j'$, the circle into which l_j inverts, and by l_j' the line into which $\mathcal{C}_j$ inverts, since the foot of the perpendicular from F on l_4' inverts into the point diametrically opposite to F on $\mathcal{C}_4$, clearly F and the four points diametrically opposite to F on the four circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ are concyclic on a circle, the *diametral circle*, which touches the circumcentric circle of $\mathcal{C}$ at F and has twice the radius. Hence, o', the orthocentric line of $\mathcal{C}'$, being twice as far from F as p', by (15),

* Davies, Mathematical Repository, vol. 6 (1835), Question 524.

† This is easily seen by inversion with F as center; compare part II.

‡ This beautiful theorem is the last of Steiner's list; the proof is that of Mention, modified to fit Johnson's notation.

§ McClelland, Geometry of the Circle, 1891, Ex. 12, p. 256; also Coolidge, Treatise on the Circle and Sphere, 1916, p. 87.

‖ Clawson, American Mathematical Monthly, vol. 24 (1917), p. 71.

inverts into the circumcentric circle of $\mathcal{C}$. Thus points on o' invert into points on $\mathcal{C}$, and points on $\mathcal{C}'$ invert into points on o. Throughout this part P' means the point inverse to P.

The following lemma will be needed:

(A) If the straight line AB be bisected at M, the inverse of M is the intersection of the circle $OA'B'$ with a circle of Apollonius of the triangle $OA'B'$ (the locus of a point P moving so that $A'P/PB' = A'O/OB'$). The perpendicular bisector of AB inverts into the circle of Apollonius named above.

I shall call M' the *Apollonian point* of A' and B' with respect to the point O.

8. **More lines through the focal point.** The following theorems are obtained by inversion from those indicated by the notation at the end of the theorems:

(32) A circle through F touching l_1 at A_{12} cuts the circle $\mathcal{C}_1$ at F and at a point on the diagonal n'''. There are twelve such circles determining four points on each diagonal (2).

(33) The Apollonian points of pairs of opposite vertices with respect to F determine a circle which passes through F (8).

(34) If a circle through F and A_{12} touching $\mathcal{C}_1$ cuts l_2 at λ_2'; a circle through F and A_{13} touching $\mathcal{C}_1$ cuts l_3 at λ_3'; a circle through F and A_{14} touching $\mathcal{C}_1$ cuts l_4 at λ_4'; then λ_2', λ_3', λ_4' are collinear with F in a line which cuts $\mathcal{C}_1$ at the point of tangency of $\mathcal{C}_1$ with a cardioide touching the four circumcircles, having F for its pole and with the diameter of the circumcentric circle produced its own length for its axis. There are three other such lines (17c).

(35) The point in which the diameter of $\mathcal{C}_1$ through F cuts l_3 and the point in which the diameter of $\mathcal{C}_3$ through F cuts l_1 are concyclic with F and A_{13} on a circle touching the circumcentric circle at F. Five other such circles exist (23).

9. **More points on the circumcentric circle.**

(36) Circles through F, A_{34} orthogonal to $\mathcal{C}_3$, through F, A_{14} orthogonal to $\mathcal{C}_1$, through F, A_{24} orthogonal to $\mathcal{C}_2$ are concurrent at a point on the circumcentric circle. There are three other such points (15).

(37) If the circle through F, A_{12} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_2', the circle through F, A_{13} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_3' and the circle through F, A_{14} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_4', then the circles through F, λ_2' orthogonal to $\mathcal{C}_2$, through F, λ_3' orthogonal to $\mathcal{C}_3$ and through F, λ_4' orthogonal to $\mathcal{C}_4$ are concurrent at a point on the circumcentric circle. There are three other such points (18).

(38) The three circles through A_{14}, A_{23} orthogonal to the circle $FA_{14}A_{23}$,

through A_{13}, A_{24} orthogonal to the circle $FA_{13}A_{24}$ and through A_{12}, A_{34} orthogonal to the circle $FA_{12}A_{34}$ have two points in common which are both on the circumcentric circle (16).

(38*a*) The circle of (33) is orthogonal to the three circles of (38) and also to the circumcentric circle and to the diametral circle (16*a*).

(39) If circles through A_{14}, F orthogonal to $\mathcal{C}_1$ and through A_{23}, F orthogonal to $\mathcal{C}_2$ intersect at α_{12}'; if circles through A_{14}, F orthogonal to $\mathcal{C}_1$ and through A_{23}, F orthogonal to $\mathcal{C}_3$ intersect at α_{13}'; if circles through A_{14}, F orthogonal to $\mathcal{C}_4$ and through A_{23}, F orthogonal to $\mathcal{C}_2$ intersect at α_{24}'; and if circles through A_{14}, F orthogonal to $\mathcal{C}_4$ and through A_{23}, F orthogonal to $\mathcal{C}_3$ intersect at α_{34}'; then the circles $F\alpha_{12}'A_{12}$, $F\alpha_{13}'A_{13}$, $F\alpha_{24}'A_{24}$, $F\alpha_{34}'A_{34}$ are concurrent at a point ζ_1 on the circumcentric circle. There are three other such points (20).

(39*a*) If the circle through A_{14}, F orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ at λ_1^{iv}, the circle through A_{14}, F orthogonal to $\mathcal{C}_4$ cuts $\mathcal{C}_4$ at λ_4', the circle through A_{23}, F orthogonal to $\mathcal{C}_2$ cuts $\mathcal{C}_2$ at λ_2''' and the circle through A_{23}, F orthogonal to $\mathcal{C}_3$ cuts $\mathcal{C}_3$ at λ_3'', then the circles $F\lambda_1^{iv}\lambda_4'$ and $F\lambda_2'''\lambda_3''$ are also concurrent at ζ_1 (20*a*).

(40) Circles through F and the Apollonian point of A_{14} and A_{24} orthogonal to $\mathcal{C}_4$ and through F and the Apollonian point of A_{13} and A_{23} orthogonal to $\mathcal{C}_3$ are concurrent at a point on the circumcentric circle. There are five other such points (21).

10. **More points on the orthocentric line.**

(41) If o_1, o_2, o_3, o_4 denote the points of intersection of o with perpendiculars from F to the sides l_1, l_2, l_3, l_4 respectively, the circles of Apollonius through F of the three triangles $FA_{12}A_{13}$, $FA_{12}A_{14}$ and $FA_{13}A_{14}$ intersect at o_1 (4).

(42) If ϵ_1, ϵ_2, ϵ_3, ϵ_4 denote the points of intersection of o with l_1, l_2, l_3, l_4, the circles Fo_2A_{12}, Fo_3A_{13} and Fo_4A_{14} intersect at ϵ_1 (5).

(42*a*) Circles through F, A_{12} tangent to l_2, through F, A_{13} tangent to l_3 and through F, A_{14} tangent to l_4 meet at a point ϵ_1' on l_1, and

$$\epsilon_1 F \epsilon_1' = \pi/2 \ (5a).$$

(43) The circles of Apollonius through F of triangles $FA_{13}A_{14}$ and $FA_{23}A_{24}$ intersect at a point τ_{34} on o. There are five other such points (6).

(44) The line $F\tau_{34}$ is perpendicular to FA_{34} (24*a*).

11. **The incentric lines.** Two circles, passing through A_{34} and F, bisect the angles between $\mathcal{C}_1$ and $\mathcal{C}_2$. Call the one whose center, C_{12}, is so placed that FC_{12} bisects C_1FC_2, $\mathcal{C}_{12}$; call the one whose center, C_{21}, is so placed that FC_{21} bisects C_2FC_1, $\mathcal{C}_{21}$. Now $\mathcal{C}_{34}$, $\mathcal{C}_{23}$, $\mathcal{C}_{42}$; $\mathcal{C}_{34}$, $\mathcal{C}_{32}$, $\mathcal{C}_{24}$; $\mathcal{C}_{43}$, $\mathcal{C}_{23}$, $\mathcal{C}_{24}$; $\mathcal{C}_{43}$, $\mathcal{C}_{32}$, $\mathcal{C}_{42}$ are concurrent respectively meeting at ι_1, ι_{12}, ι_{13}, ι_{14}. In this way sixteen points are obtained.

(45) These points are concyclic in fours in such a way as to give two sets of four coaxal circles each. These sets are mutually orthogonal and their lines of centers are the incentric lines (27) and (27*a*).

(46) The points A_{14}, A_{23}, the intersection of $\mathcal{C}_{14}$, $\mathcal{C}_{23}$ and the intersection of $\mathcal{C}_{41}$, $\mathcal{C}_{32}$; A_{13}, A_{24}, the intersection of $\mathcal{C}_{13}$, $\mathcal{C}_{24}$, and the intersection of $\mathcal{C}_{31}$, $\mathcal{C}_{42}$; A_{12}, A_{34}, the intersection of $\mathcal{C}_{12}$, $\mathcal{C}_{34}$, and the intersection of $\mathcal{C}_{21}$, $\mathcal{C}_{43}$ determine three other circles belonging to one of these sets; also A_{14}, A_{23}, the intersection of $\mathcal{C}_{14}$, $\mathcal{C}_{32}$, and the intersection of $\mathcal{C}_{41}$, $\mathcal{C}_{23}$; and two other such sets of points determine three circles belonging to the other set (27*b*).

Again, if $\mathcal{C}_{14}$, $\mathcal{C}_{23}$ intersect at κ',

$$A_{14}\kappa'A_{23} = A_{14}\kappa'F + F\kappa'A_{23} = \tfrac{1}{2}A_{14}C_{23}F + \tfrac{1}{2}FC_{14}A_{23};$$

but

$$\begin{aligned}\tfrac{1}{2}A_{14}C_{23}F = (\pi/2) - C_{23}FA_{14} = (\pi/2) - (C_{23}FC_3 + C_3FA_{14}) \\ = -\tfrac{1}{2}C_2FC_3 + A_{14}A_{24}F,\end{aligned}$$

on considering points on the circles $\mathcal{C}_{23}$, $\mathcal{C}_3$; and

$$\begin{aligned}\tfrac{1}{2}FC_{14}A_{23} = (\pi/2) - A_{23}FC_{14} = (\pi/2) - (A_{23}FC_1 + C_1FC_{14}) \\ = -\tfrac{1}{2}C_1FC_4 + FA_{24}A_{23},\end{aligned}$$

on considering points on the circles $\mathcal{C}_{14}$, $\mathcal{C}_1$; hence

$$\begin{aligned}A_{14}\kappa'A_{23} &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(C_3C_1C_2 + C_4C_2C_1) \\ &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(A_{24}FA_{34} + A_{13}FA_{34}) \\ &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(A_{24}A_{23}A_{34} + A_{13}A_{14}A_{34}) = A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}.\end{aligned}$$

But it was shown, in part I, § 6, that $A_{14}K_{22}'A_{23}$ is equal to this same expression. Hence κ' lies in the same circle with A_{14}, K_{22}' and A_{23}; hence

(47) The circles of (46) are identical with those of (27*b*); and hence the circles of (45) are coaxal with the circles of (27).

12. **Summary.** There have now been enumerated sixteen straight lines passing through the focal point: four in (17*a*), six in (24*a*), four in (35) and the two incentric lines.

There are one hundred and thirty-four circles of more or less interest passing through the focal point: four circumcircles, one circumcentric circle, one diametral circle, three circles in (2), six in (3), six in (12), four in (24*c*), twelve in (32), one in (33), twelve in (34), six in (35), twelve in (36), twelve in (37), twelve in (39), six in (39*a*), twelve in (40), twelve in (41), and twelve in (42).

On the circumcentric circle forty-seven points have been mentioned: four circumcenters, one focal point, four points in (5), six in (6), three in

(7), six in (24*b*), four in (24*c*), four in (36), four in (37), two in (38), three in (39), and six in (40).

On the orthocentric line are thirty-eight points: four orthocenters, two points in (16), four in (18), one in (19), three in (20), six in (21), four in (41), four in (42), four in (42*a*), and six in (43).

On the mid-diagonal line there are seven points: three mid-diagonals, three in (10), one mean center, besides an indefinitely great number of centers of conics touching the sides of the quadrilateral.

On the pedal line there are twelve points: four in (11), four in (13), four in (14).

On each of the circles determined by F and a pair of opposite vertices there are eight points: four in (2), one in (7), the focal point, and a pair of vertices.

Part III. Properties of a cyclic quadrangle.

13. **The mean center and the orthic center.** It will be shown in part IV that the polar reciprocal of the complete quadrilateral with respect to a circle having F as center is a cyclic quadrangle, that is, a group of four points on a circle. From the properties of the cyclic quadrangle, $L_1L_2L_3L_4$, properties of the quadrilateral can be obtained by reciprocation. It therefore becomes desirable to collect in this part certain theorems concerning points and lines connected with the cyclic quadrangle.

(α) In any quadrangle the joins of the mid-points of the three pairs of opposite sides are concurrent.* The point of concurrency, E, is the center of gravity of four equal masses placed at the vertices (Fig. 3).

The point E is called the *mean center* of the quadrangle.

(β) The join of each vertex to the center of gravity of the triangle determined by the other three vertices passes through E.† The join of the intersection of the diagonals L_1L_3, L_2L_4 to the center of gravity of the quadrangle also passes through E.‡

For, equal masses m at each vertex may be replaced by $2m$ at (L_1L_2) and $2m$ at (L_3L_4) and these in turn by $4m$ at the bisector of the join of these points. Again the four equal masses may be replaced by $3m$ at G_1, the center of gravity of the triangle $L_2L_3L_4$, and m at L_1; and E must lie on G_1L_1 so that G_1E is one third of EL_1. Finally $m/3$ each at L_2, L_3, L_4 may be replaced by m at G_1; hence the four masses may be replaced by equal masses at G_1, G_2, G_3, G_4. But G, the center of gravity of the quadrangle, is at the intersection of G_1G_3 and G_2G_4. Since the quadrangles $\mathcal{L}$ and $\mathcal{G}$ are homothetic with E as center, the intersection of the diagonals N'' and the point G are collinear with E.

* Catalan, Théorèmes et Problèmes, fourth edition, 1865, p. 8.

† Greiner, Archiv für Mathematik und Physik, vol. 60 (1877), p. 178.

‡ Deteuf, Nouvelles Annales, vol. 8 (1908), p. 442.

(γ) In a cyclic quadrangle perpendiculars from the mid-points of each side to the opposite sides are concurrent.*

We shall call this point the *orthic center* of the quadrangle, and denote it by H.

Denote (L_1L_2) by F_{12}. Let perpendiculars from F_{12}, F_{34} to L_3L_4, L_1L_2 respectively meet at H. Then $F_{12}H$ is parallel to $F_{34}O$, if O is the center of the circle, and $F_{12}O$ is parallel to $F_{34}H$. Hence O, E, H are collinear, and E bisects OH. Thus

(δ) The circumcenter, the mean center, and the orthic center are collinear, and $OE = EH$.†

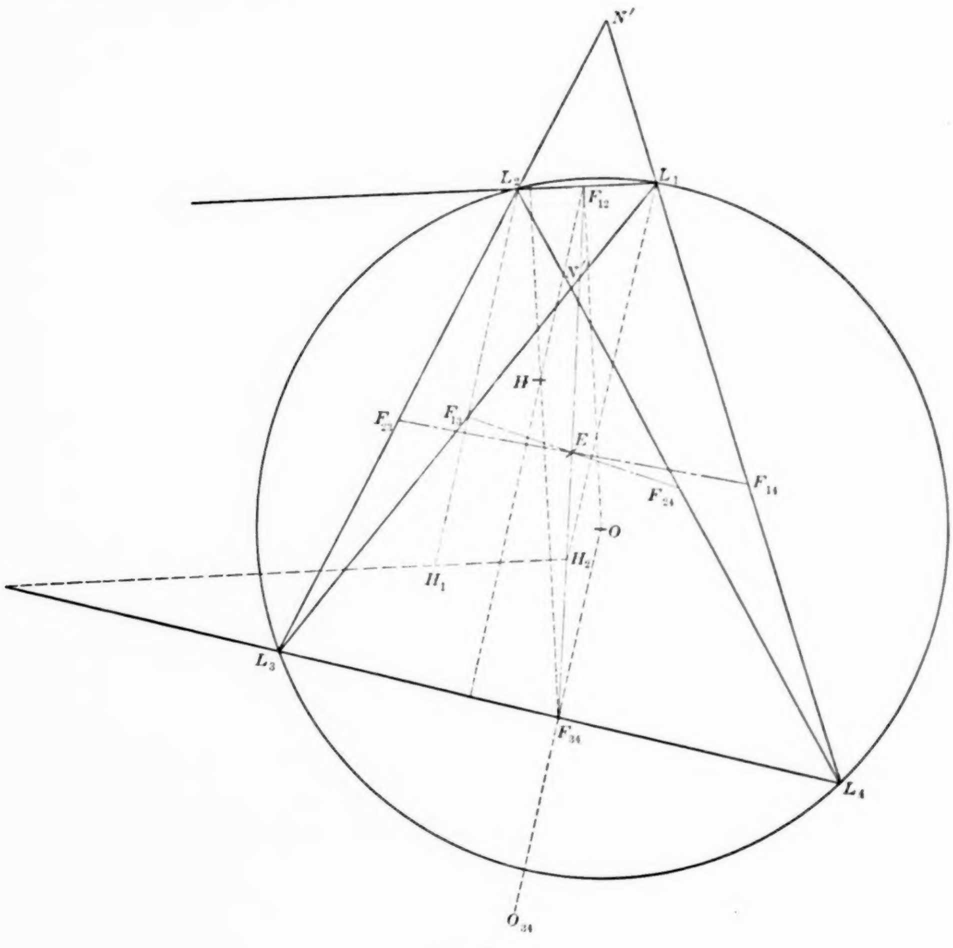

FIG. 3.

* Greiner, l. c.; Kantor, l. c., quotes Pfaff as authority for this and some other theorems in connection with this point. Nager in Monatshefte für Mathematik und Physik, Wien, vol. 7 (1896), p. 325, refers it to Euler.

† Terrier, l. c. Also Greiner and Kantor.

Denote the orthocenter of the triangle $L_2L_3L_4$ by H_1. Now

$$L_2H_1 = 2OF_{34} = L_1H_2.\text{*}$$

Hence L_1H_1 and L_2H_2 bisect each other at a point on a parallel to L_2H_1 through F_{12}, i. e., at H. Hence

(ϵ) The four lines joining each vertex to the orthocenter of the triangle determined by the other three vertices are concurrent at H.§

Again, the line joining L_1 to H_1 is bisected by the pedal line of L_1 with respect to the triangle $L_2L_3L_4$.† Hence

(ζ) The pedal line of each vertex with respect to the triangle determined by the other three vertices passes through H.‡

Further, if the image of O in L_1L_2 is O_{12}, since $F_{12}F_{34}$ is bisected at E, we have

(η) The line $O_{12}O_{34}$ is bisected at H.|| There are three such lines.

Let L_1L_3 and L_2L_4 intersect at N''. Consider the triangle $N''F_{13}F_{24}$. Two of its altitudes intersect at H. Hence on considering the third altitude, we have

(θ) The perpendicular from the intersection of two opposite sides of the quadrangle to the line joining the mid-points of those sides passes through H.|| There are three such lines.

Let the orthocenter of L_1L_2N'' be H_{12}''. Now $H_{12}''H_{23}''$ and $H_{14}''H_{34}''$ are each perpendicular to L_1L_3; also $H_{12}''H_{14}''$ and $H_{23}''H_{34}''$ are each perpendicular to L_2L_4. Hence $H_{12}''H_{34}''$ and $H_{14}''H_{23}''$ bisect each other at a point on a line through ($H_{14}''H_{34}''$) perpendicular to L_2L_4 and on a line through ($H_{12}''H_{14}''$) perpendicular to L_1L_3. But these are the lines of (γ). Hence

(ι) Six lines joining orthocenters of pairs of triangles formed by taking an intersection of two opposite sides of the quadrangle with two pairs of vertices pass through H.||

However,

(ιa) There are only three distinct lines.¶

For, for example, H_{12}'', H_{34}'', H_{13}''', H_{24}''' are collinear on the orthocentric line of the complete quadrilateral L_1L_3, L_2L_4, $N''N'''$.

(κ) The radical axes of the three pairs of circles on pairs of opposite sides as diameters pass through H.¶

For, $HF_{12} = OF_{34} = R \cos L_3L_2L_4$, if R is the radius of the circumcircle,

* Richardson and Ramsay, Modern Plane Geometry, p. 27.

† Casey, Sequel to Euclid, fourth edition, 1886, p. 36.

‡ Serret, Nouvelles Annales, vol. 7 (1848), p. 214.

§ Terrier, l. c.

|| Deteuf, l. c.

¶ Kantor, l. c.

and $HF_{34} = OF_{12} = R \cos L_1L_3L_2$. Hence

$$HF_{12}^2 - F_{12}L_2^2 = R^2(\cos^2 L_3L_2L_4 - \sin^2 L_1L_3L_2);$$

and $HF_{34}^2 - F_{34}L_3^2 = R^2(\cos^2 L_1L_3L_2 - \sin^2 L_3L_2L_4)$. But the right-hand sides of these equations are equal; hence $HF_{12}^2 - HF_{34}^2 = L_2F_{12}^2 - L_3F_{34}^2$.

Moreover, $N'L_2 \cdot N'L_3 = N'L_1 \cdot N'L_4$; $H_{12}''L_2 \cdot H_{12}''P_4'' = H_{12}''L_1 \cdot H_{12}''P_3'$; and $H_{34}''L_3 \cdot H_{34}''P_1''' = H_{34}''L_4 \cdot H_{34}''P_2^{iv}$, where P_4'' is the foot of the perpendicular from L_2 on L_1L_3. Hence N', H_{12}'', H_{34}'' lie on the radical axis of the circles with L_2L_3 and L_1L_4 as diameters.

(κa) The lines of (κ) and (ι) are identical.*

Evidently, from (ϵ), the quadrangle $H_1H_2H_3H_4$ is congruent with $L_1L_2L_3L_4$. It follows that

(λ) The join of the intersections of L_1L_2, H_3H_4; L_3L_4, H_1H_2 is bisected at H; similarly the joins of intersections of L_1L_3, H_2H_4; L_2L_4, H_1H_3 and the joins of intersections of L_1L_4, H_2H_3; L_2L_3, H_1H_4 are bisected at H.*

Kantor also proved that

(μ) All these intersections are collinear on the radical axis of the original circle and the circle circumscribing $H_1H_2H_3H_4$.*

This makes twenty-five straight lines meeting at H.†

Facts not so important for our present purpose are that the nine-point circles of the four triangles $L_2L_3L_4$,‡ $\cdots$; and of the four triangles $H_2H_3H_4$,§ $\cdots$ pass through H. The circle $N'N''N'''$ also goes through H.* Many other theorems related to this figure are given in the papers referred to in the foot notes, especially by Kantor.*

14. **The quadrilateral touching the circle at $L_1L_2L_3L_4$.** It follows at once from Pascal's theorem, considering the hexagon $L_1L_1L_3L_2L_2L_4$, that the intersection of tangents at L_1 and L_2 is collinear with N' and N''. Similarly considering $L_3L_3L_2L_4L_4L_1$, the intersection of tangents at L_3 and L_4 is collinear with N' and N''. Thus it appears that

(ν) The diagonal point triangle of the quadrangle $L_1L_2L_3L_4$ is the same as the diagonal triangle of the quadrilateral formed by the tangents to the circle at these points.‖

15. **Bisectors of the angles.** Denote the in-center and the three ex-centers of the triangle $L_2L_3L_4$ by I_1, I_{12}, I_{13}, I_{14}. Denote the mid-points of the arcs L_1L_2 by D_{12}, D_{21}, $- D_{12}$ being on the bisector of $L_1L_3L_2$ (Fig. 4).

Now D_{12} is the center of a circle passing through L_1, I_3, I_4, L_2, I_{34}, I_{43}.

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* Kantor, Sitzungsberichte der Akademie, Wien, 1878, pp. 774–792; 1879, pp. 172–192; 757–763.

† Deteuf, l. c., mentions twenty-four such lines; he says twenty-seven, but see (ιa). Cikot, Nieuw Archief voor Wiskunde, Amsterdam, vol. 6 (1905), p. 63, mentions eighteen of them.

‡ Greiner, l. c.

§ Cikot, l. c.

‖ Catalan, Théorèmes et Problèmes, 1865, pp. 83, 95.

Hence $I_3I_4I_{34}I_{43}$ is a rectangle. Its sides I_3I_4, $I_{34}I_{43}$ are perpendicular to the bisector of $L_3D_{12}L_4$; that is, to $D_{12}D_{34}$. In fact the sides of the rectangle

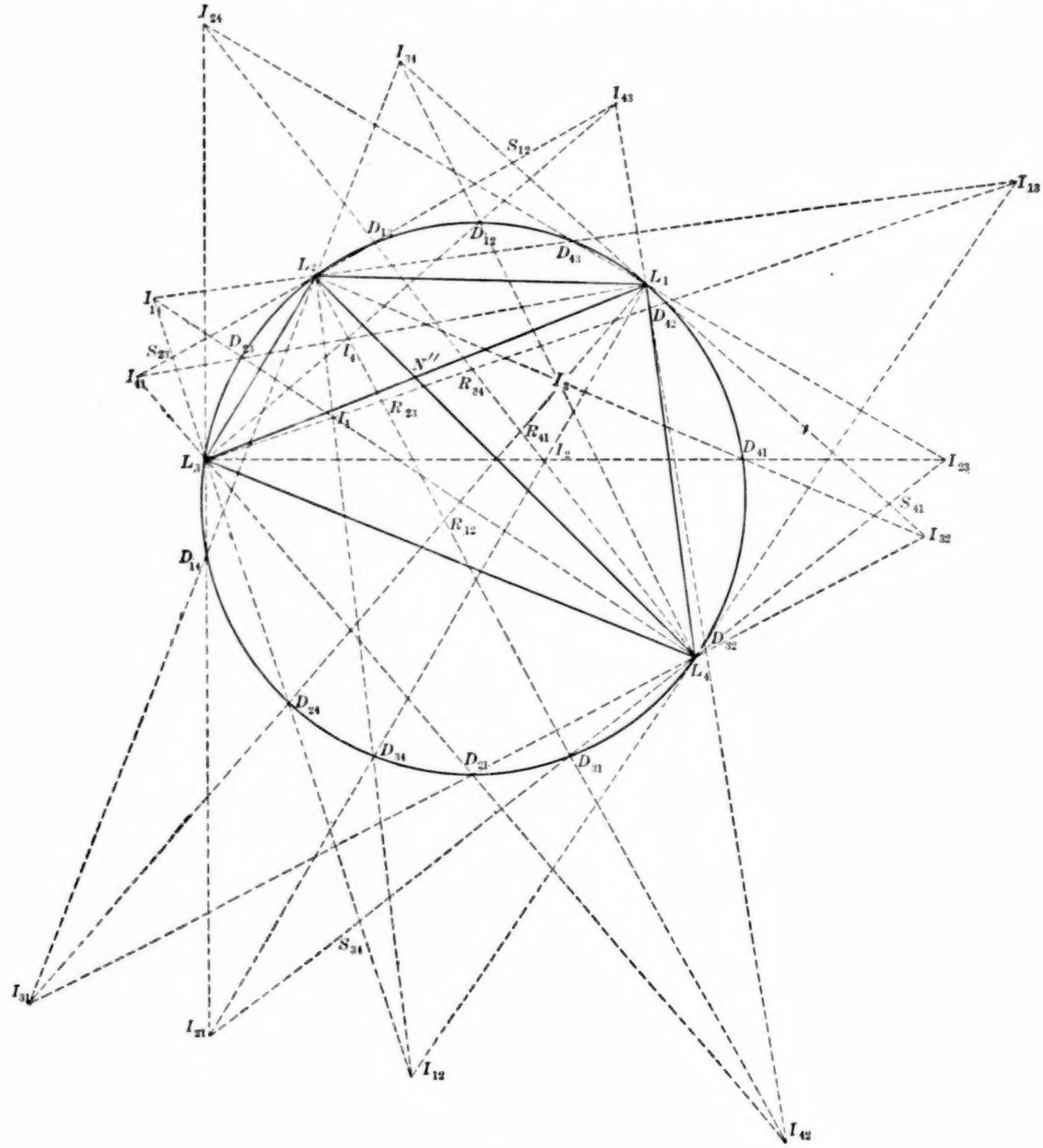

FIG. 4.

$I_3I_4I_{34}I_{43}$ are parallel to $D_{14}D_{32}$ and $D_{12}D_{34}$. But $D_{12}D_{34}$ is parallel to the bisector of $L_3N''L_4$; for, if $N''X$ bisects $L_3N''L_4$,

$$L_3N''X = \tfrac{1}{2}L_3N''L_4 = \tfrac{1}{2}L_3L_1L_4 + \tfrac{1}{2}L_1L_4L_2;$$

and, if $D_{12}D_{34}$ cuts L_1L_3 at Y,

$$L_3YD_{34} = L_3D_{12}D_{34} + L_1L_3D_{12} = \tfrac{1}{2}L_3L_1L_4 + \tfrac{1}{2}L_1L_3L_2.$$

Similarly $I_2I_3I_{23}I_{32}$, $I_1I_2I_{12}I_{21}$, $I_1I_4I_{14}I_{41}$, are rectangles whose sides are parallel to the same lines. Thus

(o) The centers of the sixteen in- and ex-circles are the intersections of four lines parallel to the bisector of the acute angle between the diagonals of the quadrangle and of four lines parallel to the bisector of the supplementary angle.*

Let L_1I_3 and L_2I_4 meet at R_{12}; let L_1I_{34} and L_2I_{43} meet at S_{12}. Now

$$L_2R_{12}L_1 = L_2L_1D_{24} + D_{31}L_2L_1 = \tfrac{1}{2}(L_2L_1L_4 + L_3L_2L_1);$$

and

$$L_3R_{34}L_4 = D_{42}L_3L_4 + L_3L_4D_{13} = \tfrac{1}{2}(L_2L_3L_4 + L_3L_4L_1).$$

But the right-hand sides are equal; hence

(π) The points R_{12}, R_{23}, R_{34}, R_{41} are concyclic. Similarly, S_{12}, S_{23}, S_{34}, S_{41} are concyclic.†

Now evidently $D_{24}D_{13}D_{42}D_{31}$ is a rectangle, for its diagonals are equal and bisect each other. But $D_{13}S_{12}D_{42} = D_{42}R_{34}D_{13} = \tfrac{1}{2}(L_2L_1L_4 + L_3L_2L_1)$; and

$$D_{13}D_{42}R_{34} = \frac{S_{12}D_{42}R_{34}}{2} = \tfrac{1}{2}L_1L_4L_3.$$

Hence the triangles $D_{42}S_{12}D_{13}$ and $D_{42}R_{34}D_{13}$ are congruent; and $S_{12}R_{34}$ is perpendicular to $D_{13}D_{42}$. But $R_{12}S_{12}L_1 = R_{12}L_2L_1 = \tfrac{1}{2}L_3L_2L_1$; and $D_{13}D_{24}L_1 = \tfrac{1}{2}L_3L_2L_1$. Hence $R_{12}S_{12}$ is parallel to $D_{13}D_{24}$. Hence S_{12}, R_{34}, R_{12} are collinear in a line parallel to $D_{13}D_{24}$. In fact

(ρ) The lines $D_{13}D_{24}$, $S_{12}R_{34}R_{12}S_{34}$, $D_{42}D_{31}$ are parallel; and $D_{24}D_{31}$, $S_{23}R_{23}R_{41}S_{41}$, $D_{13}D_{42}$ are parallel lines perpendicular to the first set.*

It is easily seen that $D_{13}D_{24}$ is parallel to $D_{12}D_{34}$; hence

(σ) The sets of parallel lines in (o) are parallel to those in (ρ).

If B_{12} bisects $S_{12}R_{12}$, $B_{12}L_2 = B_{12}L_1$. But $OL_2 = OL_1$. Hence OB_{12} is perpendicular to L_2L_1. Let the center of the circle circumscribing $R_{12}R_{23}R_{34}R_{41}$ be O_r, and the center of the circle circumscribing $S_{12}S_{23}S_{34}S_{41}$ be O_s. Then

$$S_{23}S_{12}O_s = (\pi/2) - S_{12}S_{41}S_{23} = (\pi/2) - S_{12}D_{42}D_{13}$$
$$= (\pi/2) - \tfrac{1}{2}L_1L_4L_3 = \tfrac{1}{2}L_1L_2L_3;$$

* This proof is given by Neuberg, Mathesis, vol. 6 (1906), p. 14. He refers to earlier statements of the theorem. For different proofs see American Mathematical Monthly, vol. 24 (1917), p. 429 and vol. 25 (1918), p. 241.

† Nager, l. c.

but $S_{12}L_2L_1 = \pi/2 - \frac{1}{2}L_1L_2L_3$. Hence $S_{12}O_s$ is perpendicular to L_2L_1. Similarly $R_{12}O_r$ is perpendicular to L_2L_1. Since similarly $S_{23}O_s$, $R_{23}O_r$ and $B_{23}O$ are perpendicular to L_2L_3, it follows that

(τ) The points O_r, O_s, O are collinear; and O bisects O_rO_s.*

Evidently by considering $L_1L_3L_2L_4$ as the inscribed quadrangle, we can find points R_{32}', R_{41}', R_{24}', R_{13}'; S_{32}', S_{41}', S_{24}', S_{13}'; and by considering $L_1L_2L_4L_3$ as the inscribed quadrangle we can find points R_{12}'', R_{43}'', R_{24}'', R_{13}''; S_{12}'', S_{43}'', S_{24}'', S_{13}'' for which similar statements can be made.

It follows from (o) and the consideration of the quadrangles of (τ) that

(v) The bisectors of the angles N' and N''' are respectively parallel to those of N''.†

Thus we have two sets of sixteen parallel lines each, six chords like $D_{12}D_{34}$, four lines determined by in- and ex-centers, three lines like $S_{12}R_{34}R_{12}S_{34}$, and the three bisectors of the angles formed by opposite sides of the quadrangle.

Part IV. A polar reciprocation.

16. Take any point, F, on the circumference of a circle, center O, passing through L_1, L_2, L_3, L_4. Draw any circle with F as center. The polar reciprocal of the first circle with respect to the second is the parabola, focus F, which touches l_1, l_2, l_3, l_4, the polars of L_1, L_2, L_3, L_4. Thus, by (17), we see that the cyclic quadrangle reciprocates into a complete quadrilateral, F being the quadrilateral's focal point. O, the center of the circle, evidently has for its reciprocal the orthocentric line, o, of the quadrilateral, the directrix of the parabola.

As noted in part III, properties of the cyclic quadrangle give rise to properties of the complete quadrilateral. The converse statement is true, but it is not our purpose further to develop that fact. We shall use the following lemmas:

(B) (L_1L_2) has for its reciprocal the straight line which forms a harmonic pencil with l_1, l_2 and the line joining the intersection of l_1 and l_2 with F. This line will be called the harmonic conjugate of $A_{12}F$.

(Ba) (L_1L_2) has for its reciprocal the symmedian line‡ through A_{12} of the triangle formed by l_1, l_2 and any line perpendicular to the line joining the feet of the perpendiculars from F on l_1, l_2.

If the triangle $L_1L_2L_3$ is reciprocated, a triangle $l_1l_2l_3$ results, whose vertices are A_{12}, A_{23}, A_{13}. This triangle is concyclic with F. Let the bisectors of the arcs $A_{12}A_{13}$ be U_{23}, U_{32}, $\cdots$ U_{23} lying on the bisector of $A_{12}A_{23}A_{13}$. Let FU_{23} cut $A_{12}A_{13}$ in V_{23}. Considering the Pascal hexagons $A_{23}U_{32}FU_{12}A_{12}A_{13}$ and $A_{23}U_{32}FU_{13}A_{13}A_{12}$, it follows that

* Nager, l. c.

† This theorem is new.

‡ A symmedian line is a line isogonally conjugate to a median.

(*C*) The points V_{32}, V_{13}, V_{12} and I_4 are collinear, where I_4 is the incenter of the triangle $l_1l_2l_3$.*

It is easily seen that there are four of these "inial lines" belonging to the triangle $l_1l_2l_3$, one passing through the in- and one through each of the ex-centers of that triangle. V_{13}, V_{12}, V_{32} are the reciprocals of the bisectors of $L_1L_2L_3$, $L_2L_3L_1$, $L_3L_1L_2$ respectively; and the inial lines of $l_1l_2l_3$ are the reciprocals of the in- and ex-centers of the triangle $L_1L_2L_3$.

The following theorems are obtained by reciprocation from those indicated by the notation at the end of the theorems.

(48) The harmonically conjugate lines of $A_{12}F$, $A_{34}F$ with respect to l_1, l_2 and l_3, l_4 respectively intersect at a point which is collinear with the two similarly located points in a line e (α).

Otherwise, using (*Ba*),

(48*a*) The symmedian lines through A_{12}, A_{34} in the triangles l_1l_2m, l_3l_4m, where m is the mid-diagonal line, intersect at a point which is collinear with the two similarly located points in a line e (α).

(49) The intersection of l_2 with the harmonically conjugate line of $A_{34}F$ with respect to l_3, l_4, the intersection of l_3 with the harmonically conjugate line of $A_{24}F$ with respect to l_2, l_4 and the intersection of l_4 with the harmonically conjugate line of $A_{23}F$ with respect to l_2, l_3 are collinear on a line g_1, which intersects l_1 at a point on e. Three other such points on e are similarly found (β).

(50) The intersection of the line through F perpendicular to FA_{12} with the line through A_{34} harmonically conjugate to FA_{34} and the five other such intersections are collinear on a line, h (γ).

(51) The lines h and e intersect at a point on o (δ).

(52) If h cuts l_1 at H', H' also lies on the line h_1 determined by the intersection of l_4 with a perpendicular from F to FA_{23}, the intersection of l_2 with a perpendicular from F to FA_{34} and the intersection of l_3 with a perpendicular from F to FA_{24}. There are three other such points (ϵ).

(53) If a line through F perpendicular to FA_{12} cuts l_4 at M_3, a line through F perpendicular to FA_{23} cuts l_4 at M_1, and a line through F perpendicular to FA_{13} cuts l_4 at M_2, then $A_{12}M_3$, $A_{23}M_1$, $A_{13}M_2$ are concurrent at a point on h. There are three other such points (ζ).

(54) If a line from F perpendicular to FA_{12} cuts o at M_{12} and a line o_{12} is drawn so that o, $M_{12}A_{12}$, o_{12}, $M_{12}F$ form a harmonic pencil, and o_{34}, o_{13}, $\cdots$ are similarly obtained, then o_{12}, o_{34}; o_{13}, o_{24}; o_{14}, o_{23} intersect at three points on h (η).

(55) The perpendicular from F to the line joining F to the intersection

* For a generalization of this theorem, see Clawson, American Mathematical Monthly, vol. 26 (1919), p. 59.

of the harmonically conjugate lines through A_{12} and A_{34} cuts n'' at a point on h. There are two other such points (θ).

(56) If the intersection of the perpendicular from F to FA_{23} with l_1 and the intersection of the perpendicular from F to FA_{14} with l_2 are joined, this line meets three other lines similarly drawn at a point on h. There are two other such points (ι).

(57) The line joining the intersection of h_1 and h_2 in (52) to A_{34}, the line joining the intersection of h_3 and h_4 to A_{12} and four other such lines all concur at a point on h (μ).

(58) The diagonal point triangle of the quadrangle determined by the points of contact of the parabola π with the four sides of the quadrilateral is the same as the diagonal triangle of the quadrilateral (ν).

(59) The sixteen inial lines (C) intersect by fours in four points which lie on a line through F; they also intersect by fours in four other points which lie on a line through F perpendicular to the first one. Call these lines s' and s'' (o).

(60) The line, b_{34}, joining the point where the bisector of $A_{23}FA_{24}$ cuts l_2 with the point where the bisector of $A_{13}FA_{14}$ cuts l_1, and the line, b_{21}, joining the point where the bisector of $A_{23}FA_{13}$ cuts l_3 with the point where the bisector of $A_{24}FA_{14}$ cuts l_4 intersect in a point on s'. There are six such points on s' and six on s'' (ρ).

(61) The line joining the point where the bisector of $A_{12}FA_{23}$ cuts l_2 and the point where the bisector of $A_{14}FA_{12}$ cuts l_1, the line joining the point where the bisector of $A_{34}FA_{23}$ cuts l_3 and the point where the bisector of $A_{14}FA_{34}$ cuts l_4 and two lines determined in the same way by the bisectors of the oppositely directed angles are concurrent at a point on s'. There are three such points on s' and three on s'' (σ).

(62) The points where the bisectors of $A_{12}FA_{34}$ and $A_{34}FA_{12}$ cut $A_{12}A_{34}$ are on s' and s'' respectively (υ).

Hence, from (31), it follows that

(63) The lines s' and s'' are identical with s_1 and s_2, the incentric lines.

Thus to the four points on each of the incentric lines mentioned by Steiner have been added three by Mention and Sancery (27*b*), four in this paper in (45), four in (59), six in (60), three in (61) and three in (62), making twenty-seven points in all on each incentric line, besides the focal point which is common to both.

Ursinus College,
Collegeville, Pa., January 1, 1918.

Note.* The Editors of the Annals have just called to my attention a book, Premiers Éléments d'une Théorie du Quadrilatère Cômplet, by A.

* This book was received for review by the Annals after the article of Mr. Clawson was in type, and a year and a third after it was submitted to the Annals for publication.—Ed.

Oppermann (Gauthier-Villars, 1919), which had just been received. Its author shows the same purpose exhibited in Ripert's paper and in mine—to assemble the important properties of the quadrilateral; like the present writer and unlike Ripert, M. Oppermann confines himself to the methods of pure geometry; but he covers a wider field than the writer in a less thorough way. There is some new material in the book, which covers sixty-two pages and is followed by a bibliographic supplement of ten pages more.

Most notable of the results not given in my paper are the last two theorems, XIX and XX. These theorems follow naturally after (16*a*) in part I above.

Oppermann proves that:

(16*b*) The circles of Monge* of the conics inscribed to the quadrilateral are coaxal with the circles having the diagonals as diameters (XX).†

He also proves that:

(16*c*) If three circles are drawn having for centers the intersections of the diagonals of the quadrilateral, and having for centers of similitude the six vertices of the quadrilateral (and this can be done in an infinite number of ways), the circle having for its center the radical center of the three circles and cutting them orthogonally, is coaxal with the four conjugate circles‡ of the quadrilateral (XIX).

The circles of (16*b*) and (16*c*) form two orthogonal systems of coaxal circles§—"*faisceaux conjugées de cercles*" (V).

Oppermann's theorem III is given by Ripert; but the proof is new:

(10*a*) Each of the triangles formed by three sides of the quadrilateral is crossed by the fourth side as a transversal. If on each side of the triangle a point is taken equally distant from the middle point of that side with the point where the transversal crosses that side, these three points lie on a second transversal of the triangle which is parallel to the mid-diagonal line. There are four such parallel lines (III).

The author gives my theorem (18) as an unpublished discovery of Goormaghtigh.

Besides this material, Oppermann gives in a different order about half of the theorems of part I of my paper, for the most part the classical theorems of Steiner's list and a few of Ripert's. Two of Ripert's theorems

* The *circle of Monge* of a conic is the locus of the intersection of perpendicular tangents to the conic. Kantor (see below) calls it the *Rechtkreis*. It is often called the *director circle*.

† This theorem will be found in Kantor, l. c., vol. 76 (1878), p. 774; also in Salmon, Conic Sections, Sixth Edition, 1879, p. 277.

‡ The *conjugate* (or *polar*) *circle* of a triangle is the circle with respect to which the triangle is self-conjugate (or self-polar). Its center is the orthocenter of the triangle, and it is real only for obtuse-angled triangles.

§ Kantor, l. c.

which he gives are related to theorem (9), but are not given above. Oppermann's proof of (30) is especially interesting and is somewhat shorter than that of Mention. The author does not always refer his theorems to their original sources, but his bibliographic supplement is reasonably complete, at least with respect to the French journals.

The book also contains an interesting classification of complete quadrilaterals, a test for similarity of quadrilaterals, a few properties of special types of quadrilaterals (a subject which has a fairly extensive literature of its own), and some other topics of minor interest.

The reader who wishes to pursue this subject further should not neglect Oppermann's book, nor the articles of Ripert, Terrier and Sancery cited above. Each of these papers extends the subject in its own special direction.

May 31, 1919.

TRIPLY CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS.

By Luther Pfahler Eisenhart.

1. When the Cartesian coördinates x_1, x_2, x_3 of a point M in space are functions of three parameters u_1, u_2, u_3 which are functionally independent and satisfy three equations of the form

$$
\begin{aligned}
\frac{\partial^2\theta}{\partial u_1\partial u_2} &= b_{12}\frac{\partial\theta}{\partial u_1} + b_{21}\frac{\partial\theta}{\partial u_2},\\
\frac{\partial^2\theta}{\partial u_2\partial u_3} &= b_{23}\frac{\partial\theta}{\partial u_2} + b_{32}\frac{\partial\theta}{\partial u_3},\\
\frac{\partial^2\theta}{\partial u_3\partial u_1} &= b_{31}\frac{\partial\theta}{\partial u_3} + b_{13}\frac{\partial\theta}{\partial u_1},
\end{aligned}
\tag{1}
$$

where the functions b_{ij} ordinarily are functions of u_1, u_2, u_3, the parametric surfaces u_i = const. ($i = 1, 2, 3$) cut along conjugate systems of curves. Such a system is called a *triply conjugate system*.

The functions b_{ij} are not arbitrary, but are subject to certain conditions, which we shall now determine. When the two expressions for $\partial^3\theta/\partial u_1\partial u_2\partial u_3$ obtained by differentiating the first two of equations (1) are equated, we get

$$
\frac{\partial\theta}{\partial u_1}\left(\frac{\partial b_{12}}{\partial u_3} + b_{12}b_{13} - b_{12}b_{23} - b_{32}b_{13}\right) + \frac{\partial\theta}{\partial u_2}\left(\frac{\partial b_{21}}{\partial u_3} - \frac{\partial b_{23}}{\partial u_1}\right)
$$
$$
- \frac{\partial\theta}{\partial u_3}\left(\frac{\partial b_{32}}{\partial u_1} + b_{32}b_{31} - b_{12}b_{31} - b_{21}b_{13}\right) = 0.
$$

Since, by hypothesis, this condition must be satisfied by three functionally independent solutions of (1), the expressions in parentheses must be equal to zero. Proceeding in like manner with the first and third of equations (1) and also with the second and third, we obtain similar conditions all of which may be written:

$$
\frac{\partial b_{ij}}{\partial u_k} = \frac{\partial b_{ik}}{\partial u_j}, \tag{2}
$$

$$
\frac{\partial b_{ij}}{\partial u_k} + b_{ij}b_{ik} - b_{ij}b_{jk} - b_{kj}b_{ik} = 0, \tag{3}
$$

where the subscripts i, j, k are different from one another and take on the values 1, 2, 3.

The first three of these equations are satisfied, if we put

$$b_{ij} = \frac{\partial}{\partial u_j} \log a_i, \tag{4}$$

thus defining three functions a_i, to within a multiple which is a function of u_i alone. When these expressions are substituted in (3), the latter reduce to the three equations

$$\frac{\partial^2 a_i}{\partial u_j \partial u_k} = \frac{\partial \log a_j}{\partial u_k} \frac{\partial a_i}{\partial u_j} + \frac{\partial \log a_k}{\partial u_j} \frac{\partial a_i}{\partial u_k}. \tag{5}$$

Now equations (1) become

$$\frac{\partial^2 \theta}{\partial u_i \partial u_j} = \frac{\partial \log a_i}{\partial u_j} \frac{\partial \theta}{\partial u_i} + \frac{\partial \log a_j}{\partial u_i} \frac{\partial \theta}{\partial u_j}. \tag{6}$$

From the preceding discussion it follows that, when any three functions a_i satisfy (5), the corresponding equations (6) are compatible and admit solutions not functionally expressible in terms of two of them. The general solution involves three arbitrary functions of a single variable.*

2. We are concerned in this paper with the triply conjugate systems with equal invariants, that is those for which the functions a_i satisfy the conditions

$$\frac{\partial^2 \log a_i}{\partial u_i \partial u_j} = \frac{\partial^2 \log a_j}{\partial u_i \partial u_j}.$$

These equations may be replaced by

$$\frac{a_1}{a_2} = \frac{\varphi_2}{\psi_1}, \qquad \frac{a_2}{a_3} = \frac{\varphi_3}{\psi_2}, \qquad \frac{a_3}{a_1} = \frac{\varphi_1}{\psi_3}, \tag{7}$$

where a function φ_i or ψ_i is independent of u_i. Multiplying these equations together, we get

$$\frac{\varphi_1}{\psi_1} \cdot \frac{\varphi_2}{\psi_2} \cdot \frac{\varphi_3}{\psi_3} = 1.$$

If we take the logarithmic derivative of this equation with respect to u_2 and u_3, we obtain

$$\frac{\partial^2}{\partial u_2 \partial u_3} \log \frac{\varphi_1}{\psi_1} = 0,$$

that is φ_1/ψ_1 is equal to the ratio of a function of u_3 alone and a function of u_2 alone. Since similar results are true for φ_2/ψ_2 and φ_3/ψ_3, it is readily shown that in all generality the above equation may be replaced by

$$\frac{\varphi_1}{\psi_1} = \frac{U_3}{U_2}, \qquad \frac{\varphi_2}{\psi_2} = \frac{U_1}{U_3}, \qquad \frac{\varphi_3}{\psi_3} = \frac{U_2}{U_1},$$

* Darboux, Leçons sur la théorie générale des surfaces, vol. 4, p. 271.

where U_i denotes a function of u_i alone. When these values are substituted in (7) and we put

$$e^{\sigma_1} = \varphi_1 U_2, \qquad e^{\sigma_2} = \varphi_2 U_3, \qquad e^{\sigma_3} = \varphi_3 U_1,$$

we may replace (7) by

$$\text{(8)} \qquad a_1 = \frac{e^{\sigma_2+\sigma_3}}{\mu}, \qquad a_2 = \frac{e^{\sigma_3+\sigma_1}}{\mu}, \qquad a_3 = \frac{e^{\sigma_1+\sigma_2}}{\mu},$$

where μ is in general a function of all three parameters and σ_i is independent of u_i.

When the expressions (8) are substituted in (5), we find that μ must satisfy the three equations

$$\text{(9)} \qquad \frac{\partial^2\varphi}{\partial u_i \partial u_j} = \frac{\partial\sigma_k}{\partial u_j}\frac{\partial\varphi}{\partial u_i} + \frac{\partial\sigma_k}{\partial u_i}\frac{\partial\varphi}{\partial u_j} + \left(\frac{\partial\sigma_j}{\partial u_i}\frac{\partial\sigma_i}{\partial u_j} - \frac{\partial\sigma_k}{\partial u_j}\frac{\partial\sigma_j}{\partial u_i} - \frac{\partial\sigma_k}{\partial u_i}\frac{\partial\sigma_i}{\partial u_j}\right)\varphi.$$

If the expressions (8) are substituted in (6), and at the same time we put

$$\text{(10)} \qquad \theta = \varphi/\mu,$$

the equations for φ reduce to (9). Hence the problem of finding triply conjugate systems with equal invariants reduces to the integration of equations (9). In fact, if θ_1, θ_2, θ_3, θ_4 are four independent solutions of (9), then

$$\text{(11)} \qquad x_1 = \frac{\theta_1}{\theta_4}, \qquad x_2 = \frac{\theta_2}{\theta_4}, \qquad x_3 = \frac{\theta_3}{\theta_4},$$

are the Cartesian coördinates of space referred to such a triply conjugate system with equal point invariants.

3. We consider now the conditions which the three functions σ_i must satisfy in order that (9) may admit four functionally independent solutions.

In the first place we observe that if we have three functions a_i given by (8) which satisfy the conditions of the problem, the same system of equations is given by the functions σ_i' and μ', defined by

$$\text{(12)} \qquad \sigma_i' = \sigma_i + U_j + U_k, \qquad \mu' = \mu e^{U_1+U_2+U_3},$$

where U_i is a function of u_i alone. In fact,

$$a_i' = \frac{e^{\sigma_j'+\sigma_k'}}{\mu'} = e^{U_i} a_i.$$

But in § 1 we observed that a_i was determined only to within a factor of u_i alone. This establishes the result.

We turn now to the determination of the conditions which the functions σ_i must satisfy in order that equations (9) may be consistent. When we

equate in pairs the expressions for $\partial^3\varphi/\partial u_1\partial u_2\partial u_3$ obtained from (9) by differentiation, we get the three conditions

$$A - 2B + C = 0, \qquad A + B - 2C = 0, \qquad B + C - 2A = 0,$$

where we have put

$$A = \left(\frac{\partial\sigma_2}{\partial u_1} - \frac{\partial\sigma_3}{\partial u_1}\right)\frac{\partial^2\sigma_1}{\partial u_2\partial u_3}, \qquad B = \left(\frac{\partial\sigma_3}{\partial u_2} - \frac{\partial\sigma_1}{\partial u_2}\right)\frac{\partial^2\sigma_2}{\partial u_3\partial u_1},$$

$$C = \left(\frac{\partial\sigma_1}{\partial u_3} - \frac{\partial\sigma_2}{\partial u_3}\right)\frac{\partial^2\sigma_3}{\partial u_1\partial u_2}.$$

In order that these three conditions may be satisfied simultaneously, we must have

$$A = B = C. \tag{13}$$

We consider first the case

$$A = B = C = 0. \tag{14}$$

If

$$\frac{\partial\sigma_2}{\partial u_1} - \frac{\partial\sigma_3}{\partial u_1} = 0,$$

we have on differentiating this equation separately with respect to u_2 and u_3

$$\frac{\partial^2\sigma_2}{\partial u_1\partial u_3} = 0, \qquad \frac{\partial^2\sigma_3}{\partial u_1\partial u_2} = 0,$$

in which case B and C are zero also. In order that these three equations be satisfied we must have

$$\sigma_2 = U_1 + U_3, \qquad \sigma_3 = U_1 + U_2,$$

where U_i is an arbitrary function of u_i alone. In consequence of (12), we may take $\sigma_2 = \sigma_3 = 0$, and σ_1 arbitrary. Hence a first solution is afforded by the three values

$$\sigma_i \text{ arbitrary}, \qquad \sigma_j = \sigma_k = 0. \tag{15}$$

We consider secondly the other possibility for (14) afforded by

$$\frac{\partial^2\sigma_1}{\partial u_2\partial u_3} = 0, \qquad \frac{\partial^2\sigma_2}{\partial u_3\partial u_1} = 0, \qquad \frac{\partial^2\sigma_3}{\partial u_1\partial u_2} = 0.$$

Because of (12) the general solution in this case is

$$\sigma_1 = U_2 - U_3, \qquad \sigma_2 = U_3 - U_1, \qquad \sigma_3 = U_1 - U_2, \tag{16}$$

where U_i is an arbitrary function of u_i alone, none of which may be constant; otherwise we have a type reducible to (15).

We consider finally the general case where the common value of A,

B and C is different from zero. If we put

$$\frac{\partial^2 \sigma_1}{\partial u_2 \partial u_3} = \frac{1}{h}\left(\frac{\partial \sigma_1}{\partial u_2} - \frac{\partial \sigma_3}{\partial u_2}\right)\left(\frac{\partial \sigma_2}{\partial u_3} - \frac{\partial \sigma_1}{\partial u_3}\right), \tag{17}$$

where h is to be determined, equations (14) may be replaced by

$$\begin{aligned} \frac{\partial^2 \sigma_2}{\partial u_3 \partial u_1} &= \frac{1}{h}\left(\frac{\partial \sigma_2}{\partial u_3} - \frac{\partial \sigma_1}{\partial u_3}\right)\left(\frac{\partial \sigma_3}{\partial u_1} - \frac{\partial \sigma_2}{\partial u_1}\right), \\ \frac{\partial^2 \sigma_3}{\partial u_1 \partial u_2} &= \frac{1}{h}\left(\frac{\partial \sigma_3}{\partial u_1} - \frac{\partial \sigma_2}{\partial u_1}\right)\left(\frac{\partial \sigma_1}{\partial u_2} - \frac{\partial \sigma_3}{\partial u_2}\right). \end{aligned} \tag{18}$$

When equations (17) and (18) are differentiated with respect to u_1, u_2 and u_3 respectively, it is found that h is a constant.

If in equation (17) we make the substitution

$$\sigma_1 = \sigma_2 + \sigma_3 + \tau,$$

the resulting equation is

$$h\frac{\partial^2 \tau}{\partial u_2 \partial u_3} + \frac{\partial \tau}{\partial u_2}\frac{\partial \tau}{\partial u_3} = 0,$$

of which the general integral is $\tau = h \log(\varphi_2 + \varphi_3)$, where φ_2 and φ_3 are arbitrary functions independent of u_2 and u_3 respectively. When this value of τ is put in the above equation for σ_1, we may in all generality give u_1 a constant value in the left-hand member, in which case we have

$$\sigma_1 = \bar{U}_2 + \bar{U}_3 + h \log(U_2 - U_3),$$

where $\bar{U}_i$ and U_i are functions of u_i alone.

When this value is substituted in (17), the resulting equation may be put in the form

$$\frac{hU_2'}{\dfrac{\partial \sigma_3}{\partial u_2} - \bar{U}_2'} - U_2 = \frac{hU_3'}{\dfrac{\partial \sigma_2}{\partial u_3} - \bar{U}_3'} - U_3.$$

Since the respective members of this equation are independent of u_3 and u_2, they are equal to a function of u_1 alone, say $-U_1$. Then we have by integration.

$$\begin{aligned} \sigma_2 &= \bar{U}_3 + h \log(U_3 - U_1) + \bar{U}_1, \\ \sigma_3 &= \bar{U}_2 + h \log(U_1 - U_2) + \psi(u_1), \end{aligned}$$

where $\bar{U}_1$ and ψ are functions of u_1 alone.

We remark that none of the functions U_1, U_2, U_3 may be constant. For if $U_1 = \text{const.}$, we have

$$\frac{\partial^2 \sigma_2}{\partial u_1 \partial u_3} = \frac{\partial^2 \sigma_3}{\partial u_1 \partial u_2} = 0,$$

which is contrary to hypothesis.

When the above values of σ_1, σ_2 and σ_3 are substituted in the first of (18), we get $\psi' = \overline{U}_1'$. If we write $\psi = \overline{U}_1 + 2a$, and then replace $\overline{U}_1$, $\overline{U}_2$, $\overline{U}_3$ by $\overline{U}_1 - a$, $\overline{U}_2 - a$, $\overline{U}_3 + a$ respectively, we get the same result as if we had taken $a = 0$; this therefore will be done. In view of this result and (12) we have in all generality

$$(19) \qquad \sigma_1 = h \log (U_2 - U_3), \qquad \sigma_2 = h \log (U_3 - U_1), \qquad \sigma_3 = h \log (U_1 - U_2),$$

where none of the functions U_i is constant.*

4. We consider first the case given by (15). Equations (9) reduce to

$$(20) \qquad \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} = 0, \qquad \frac{\partial^2 \varphi}{\partial u_3 \partial u_1} = 0, \qquad \frac{\partial^2 \varphi}{\partial u_2 \partial u_3} = \frac{\partial \sigma_1}{\partial u_3}\frac{\partial \varphi}{\partial u_2} + \frac{\partial \sigma_1}{\partial u_2}\frac{\partial \varphi}{\partial u_3}.$$

From the first two of these equations it follows that φ is the sum of a function of u_1 alone and a function independent of u_1. Hence the point coördinates of a triply conjugate system of this type are of the form

$$x = \frac{\varphi_1 + U_1}{\overline{\varphi}_1 + \overline{U}_1},$$

where φ_1 and $\overline{\varphi}_1$ are solutions of the third of equations (20) and are independent of u_1.

When in particular $\sigma_1 = 0$, we have

$$x = \frac{U_1 + U_2 + U_3}{\overline{U}_1 + \overline{U}_2 + \overline{U}_3},$$

where U_i and $\overline{U}_i$ are functions of u_i alone.

5. When the values (16) are substituted in equations (9), it is found that by a transformation of variables U_i may be replaced by u_i in all generality for $i = 1, 2, 3$, and the equations become

$$(21) \qquad \begin{aligned} &\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + \frac{\partial \varphi}{\partial u_1} - \frac{\partial \varphi}{\partial u_2} + 3\varphi = 0, \\ &\frac{\partial^2 \varphi}{\partial u_2 \partial u_3} + \frac{\partial \varphi}{\partial u_2} - \frac{\partial \varphi}{\partial u_3} + 3\varphi = 0, \\ &\frac{\partial^2 \varphi}{\partial u_3 \partial u_1} + \frac{\partial \varphi}{\partial u_3} - \frac{\partial \varphi}{\partial u_1} + 3\varphi = 0. \end{aligned}$$

We observe that if φ is any solution of this system so also is any of the derivatives of φ.

* Cf. Darboux, Lecons sur les systémes orthogonaux, second edition, pp. 231, et seq. In these pages Darboux solves the problem of finding all triply orthogonal systems such that all the surfaces are isothermic, that is, the lines of curvature on each surface form an isothermic net. These systems are triply conjugate with equal point invariants.

Gronwall in the next paper* has found the solution $\varphi(u_1, u_2, u_3)$ of these equations determined uniquely by its values $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$. If f_0 and f_1 denote Bessel's functions of order zero and one respectively, the integral is

$$\begin{aligned}\varphi(u_1, u_2, u_3) = e^{-u_2}\varphi(u_1, 0, u_3) + e^{u_1}\varphi(0, u_2, u_3) - e^{u_1-u_2}f_0(u_1u_2)\varphi(0, 0, u_3)\\ - e^{-u_2}\int_0^{u_1} e^{u_1-v_1}4u_2f_1(u_2(u_1 - v_1))\varphi(v_1, 0, u_3)dv_1\\ - e^{u_1}\int_0^{u_2} e^{v_2-u_2}4u_1f_1(u_1(u_2 - v_2))\varphi(0, v_2, u_3)dv_2,\end{aligned}$$

where $\varphi(u_1, 0, u_3)$ and $\varphi(0, u_2, u_3)$ are given by the similar equations, namely

$$\begin{aligned}\varphi(0, u_2, u_3) = e^{-u_3}\varphi(0, u_2, 0) + e^{u_2}\varphi(0, 0, u_3) - e^{u_2-u_3}f_0(u_2u_3)\varphi(0, 0, 0)\\ - e^{-u_3}\int_0^{u_2} e^{u_2-v_2}4u_3f_1(u_3(u_2 - v_2))\varphi(0, v_2, 0)dv_2\\ - e^{u_2}\int_0^{u_3} e^{v_3-u_3}4u_2f_1(u_2(u_3 - v_3))\varphi(0, 0, v_3)dv_3,\end{aligned}$$

and

$$\begin{aligned}\varphi(u_1, 0, u_3) = e^{-u_1}\varphi(0, 0, u_3) + e^{u_3}\varphi(u_1, 0, 0) - e^{u_3-u_1}f_0(u_3u_1)\varphi(0, 0, 0)\\ - e^{-u_1}\int_0^{u_3} e^{u_3-v_3}4u_1f_1(u_1(u_3 - v_3))\varphi(0, 0, v_3)dv_3\\ - e^{u_3}\int_0^{u_1} e^{v_1-u_1}4u_3f_1(u_3(u_1 - v_1))\varphi(v_1, 0, 0)dv_1.\end{aligned}$$

Gronwall† reduces this solution to other forms.

6. When the values (19) are substituted in equations (9), we find that by a transformation of the variables we may in all generality take $U_i = u_i$ $(i = 1, 2, 3)$. Then the equations become

$$\begin{aligned}\frac{\partial^2\varphi}{\partial u_1\partial u_2} &= \frac{h}{u_2 - u_1}\left(\frac{\partial\varphi}{\partial u_1} - \frac{\partial\varphi}{\partial u_2}\right),\\ \frac{\partial^2\varphi}{\partial u_2\partial u_3} &= \frac{h}{u_3 - u_2}\left(\frac{\partial\varphi}{\partial u_2} - \frac{\partial\varphi}{\partial u_3}\right),\\ \frac{\partial^2\varphi}{\partial u_3\partial u_1} &= \frac{h}{u_1 - u_3}\left(\frac{\partial\varphi}{\partial u_3} - \frac{\partial\varphi}{\partial u_1}\right).\end{aligned} \tag{22}$$

* On a system of linear partial differential equations of the hyperbolic type, pp. 273–276, of this number of the Annals.

† L. c.

We make the following observation concerning these equations. If each be differentiated with respect to u_1, u_2 and u_3, and we put

$$\varphi_1 = \frac{\partial^3 \varphi}{\partial u_1 \partial u_2 \partial u_3}$$

we find that φ_1 satisfies the equations

$$\begin{aligned}
\frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2} &= \frac{h-1}{u_2 - u_1}\left(\frac{\partial \varphi_1}{\partial u_1} - \frac{\partial \varphi_1}{\partial u_2}\right), \\
\frac{\partial^2 \varphi_1}{\partial u_2 \partial u_3} &= \frac{h-1}{u_3 - u_2}\left(\frac{\partial \varphi_1}{\partial u_2} - \frac{\partial \varphi_1}{\partial u_3}\right), \\
\frac{\partial^2 \varphi_1}{\partial u_3 \partial u_1} &= \frac{h-1}{u_1 - u_3}\left(\frac{\partial \varphi_1}{\partial u_3} - \frac{\partial \varphi_1}{\partial u_1}\right).
\end{aligned} \tag{23}$$

Hence when the general integral of the system (22) is known for a given value of h, we can find the integrals of the equations for $h - n$, where n is an integer.

Again if we put

$$\varphi = (u_1 - u_2)^h \overline{\varphi}$$

in the first of equations (23), it becomes

$$\frac{\partial^2 \overline{\varphi}}{\partial u_1 \partial u_2} = \frac{-h(h+1)}{(u_1 - u_2)^2}\overline{\varphi}.$$

From this it is seen that, when $h = -1$, the general integral of the first of equations (22) is $(u_1 - u_2)^h(\psi_1 + \psi_2)$, where ψ_1 and ψ_2 are independent of u_1 and u_2 respectively. When it is required that this expression satisfy the second and third of (22), it is found that the general integral of this system of equations for $h = -1$ is of the form

$$\varphi = \frac{U_1}{(u_1 - u_2)(u_1 - u_3)} + \frac{U_2}{(u_2 - u_3)(u_2 - u_1)} + \frac{U_3}{(u_3 - u_1)(u_3 - u_2)},$$

where U_i is an arbitrary function of u_i alone. In view of the preceding remarks we may obtain from this expression by differentiation the general integral of (22) for *any negative integral* value of h.

If in equations (6) we put

$$\begin{aligned}
a_1 &= (u_1 - u_2)^{m_2}(u_3 - u_1)^{m_3}, \\
a_2 &= (u_2 - u_3)^{m_3}(u_1 - u_2)^{m_1}, \\
a_3 &= (u_3 - u_1)^{m_1}(u_2 - u_3)^{m_2},
\end{aligned} \tag{24}$$

we get

$$\frac{\partial^2 \varphi}{\partial u_i \partial u_j} = -\frac{m_j}{u_i - u_j}\frac{\partial \theta}{\partial u_i} + \frac{m_i}{u_i - u_j}\frac{\partial \theta}{\partial u_j}. \tag{25}$$

When the transformation of Laplace, defined by

$$\varphi_1 = \varphi - \frac{a_2}{\frac{\partial a_2}{\partial u_1}} \frac{\partial \varphi}{\partial u_1}, \tag{26}$$

is applied to this system of equations, we obtain another system of the form (6) in which the quantities $\bar{a}_i$, entering in the coefficients, are of the form

$$\bar{a}_1 = (u_1 - u_2)^{m_2+1}(u_3 - u_1)^{m_3},$$
$$\bar{a}_2 = (u_2 - u_3)^{m_3}(u_1 - u_2)^{m_1-1},$$
$$\bar{a}_3 = (u_3 - u_1)^{m_1-1}(u_2 - u_3)^{m_2+1},$$

which are of the same form as (24) with m_1, m_2, m_3 replaced by $m_1 - 1$, $m_2 + 1$, m_3 respectively.

Evidently equations (22) are of the form (25) with $m_1 = m_2 = m_3 = h$. Hence if h is a positive integer and we apply the transformation (26) h times in succession, we get the system

$$\begin{aligned} \frac{\partial^2 \psi}{\partial u_1 \partial u_2} &= -\frac{2h}{u_1 - u_2}\frac{\partial \psi}{\partial u_1}, \qquad \frac{\partial^2 \psi}{\partial u_3 \partial u_1} = \frac{h}{u_3 - u_1}\frac{\partial \psi}{\partial u_1}, \\ \frac{\partial^2 \psi}{\partial u_2 \partial u_3} &= -\frac{h}{u_2 - u_3}\frac{\partial \psi}{\partial u_2} + \frac{2h}{u_2 - u_3}\frac{\partial \psi}{\partial u_3}. \end{aligned} \tag{27}$$

The general integral of the first two of these equations is

$$\psi = \int (u_1 - u_2)^{2h}(u_3 - u_1)^h U_1 du_1 + \rho_1, \tag{28}$$

where U_1 is an arbitrary function of u_1 alone, and ρ_1 is independent for u_1. When this expression for ψ is substituted in the third of (27), we find that ρ_1 must be a solution of the third. But the general integral of this equation is*

$$\rho_1 = (u_2 - u_3)^{3h+1} \frac{\partial^{3h}}{\partial u_2^{2h} \partial u_3^{h}} \left(\frac{U_2 - U_3}{u_2 - u_3} \right),$$

where U_2 and U_3 are arbitrary functions of u_2 and u_3 respectively. Hence we have the general solution of (27).

Equation (26) may be written

$$\varphi = \varphi_1 - \frac{\bar{a}_1}{\frac{\partial \bar{a}_1}{\partial u_2}} \frac{\partial \varphi_1}{\partial u_2},$$

which is the inverse of the Laplace transformation. For the present case

* Darboux, Lecons, vol. 2, p. 65.

this and the inverses of the successive transformations become

$$
(29)\qquad
\begin{aligned}
\varphi &= \varphi_1 + \frac{u_1 - u_2}{h+1}\frac{\partial \varphi_1}{\partial u_2},\\
\varphi_1 &= \varphi_2 + \frac{u_1 - u_2}{h+2}\frac{\partial \varphi_2}{\partial u_2},\\
&\cdots\cdots\cdots\\
\varphi_{h-1} &= \varphi_h + \frac{u_1 - u_2}{2h}\frac{\partial \varphi_h}{\partial u_2}.
\end{aligned}
$$

Now φ_h is the function ψ given by (28), and hence φ is readily obtained by the differentiations indicated in (29).

7. Consider now the case when $h > 0$. The definite integral

$$
(30)\qquad \int_a^{u_1} \psi(t)(t - u_1)^h(u_2 - t)^h(u_3 - t)^h dt,
$$

where ψ is an arbitrary function of t and a is a constant, is an evident generalization of the integral of the first of equations (22) given by Poisson.*

By applying the usual method of differentiating a definite integral we show that the integral (30) is a solution of equations (22) whatever be ψ.

The same is true of the integrals

$$
\int_a^{u_2} \psi(t)(u_1 - t)^h(t - u_2)^h(u_3 - t)^h dt,
$$

$$
\int_a^{u_3} \psi(t)(u_1 - t)^h(u_2 - t)^h(t - u_3)^h dt.
$$

Thus we have three independent integrals each involving an arbitrary function.

8. For the case $h < 0$ we apply to the system (22) the method of Riemann, as Gronwall has done in the case of equations (21)†, and find to within quadratures the unique integral $\varphi(u_1, u_2, u_3)$ determined by arbitrary values of $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$.

We recall that to integrate an equation

$$
\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} + a(u_1, u_2)\frac{\partial \varphi}{\partial u_1} + b(u_1, u_2)\frac{\partial \varphi}{\partial u_2} + c(u_1, u_2)\varphi = 0
$$

by the Riemann method we form the adjoint equation

$$
\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{\partial}{\partial v_1}(a(v_1, v_2)t) - \frac{\partial}{\partial v_2}(b(v_1, v_2)t) + c(v_1, v_2)t = 0,
$$

* Journal de l'Ecole Polytechnique, Cahier 19 (1823), p. 215.

† L. c.

and find a solution $t(v_1, v_2;\ u_1, u_2)$ satisfying the initial conditions

$$(31)\qquad \begin{aligned} t(v_1, u_2;\ u_1, u_2) &= e^{\int_{u_1}^{v_1} b(v_1, u_2)dv_1}, \\ t(u_1, v_2;\ u_1, u_2) &= e^{\int_{u_2}^{v_2} a(u_1, v_2)dv_2}. \end{aligned}$$

Then the integral of the given equation reducing to given values of $\varphi(0, u_2)$ and $\varphi(u_1, 0)$ is

$$(32)\qquad \begin{aligned} \varphi(u_1, u_2) = {} & t(u_1, 0;\ u_1, u_2)\varphi(u_1, 0) + t(0, u_2;\ u_1, u_2)\varphi(0, u_2) \\ & - t(0, 0;\ u_1, u_2)\varphi(0, 0) \\ & + \int_0^{u_1}\left[b(v_1, 0)t(v_1, 0;\ u_1, u_2) - \frac{\partial}{\partial v_1}t(v_1, 0;\ u_1, u_2)\right]\varphi(v_1, 0)dv_1 \\ & + \int_0^{u_2}\left[a(0, v_2)t(0, v_2;\ u_1, u_2) - \frac{\partial}{\partial v_2}t(0, v_2;\ u_1, u_2)\right]\varphi(0, v_2)dv_2. \end{aligned}$$

9. For the first of equations (22)

$$a(u_1, u_2) = \frac{h}{u_1 - u_2}, \qquad b(u_1, u_2) = \frac{h}{u_2 - u_1},$$

and the adjoint equation is

$$\frac{\partial^2 t}{\partial v_1 \partial v_2} - \frac{h}{v_1 - v_2}\frac{\partial t}{\partial v_1} + \frac{h}{v_1 - v_2}\frac{\partial t}{\partial v_2} + \frac{2h}{(v_1 - v_2)^2}t = 0.$$

In this case the initial values (31) are reducible to

$$t(v_1, u_2;\ u_1, u_2) = \left(\frac{u_1 - u_2}{v_1 - u_2}\right)^h, \qquad t(u_1, v_2;\ u_1, u_2) = \left(\frac{u_2 - u_1}{v_2 - u_1}\right)^h.$$

Darboux* has shown that the solution of the adjoint equation satisfying these conditions is

$$\begin{aligned} t(v_1, v_2;\ u_1, u_2) = {} & (u_2 - v_1)^h(v_2 - v_1)^{-2h}(v_2 - u_1)^h \\ & \times F\left(-h, -h, 1, \frac{(v_1 - u_1)\cdot(v_2 - u_2)}{(v_1 - u_2)\cdot(v_2 - u_1)}\right), \end{aligned}$$

where F denotes the hypergeometric series with the constants $-h$, $-h$, 1 and the argument $(v_1 - u_1)(v_2 - u_2)/(v_1 - u_2)(v_2 - u_1)$.

In order to obtain the expression (32), we note that

$$t(u_1, 0;\ u_1, u_2) = u_1^{-h}(u_1 - u_2)^h, \qquad t(0, u_2;\ u_1, u_2) = u_2^{-h}(u_2 - u_1)^h,$$
$$t(0, 0;\ u_1, u_2) = 0,$$

* Lecons, 2d ed., vol. 2, p. 83.

and that $F(-h, -h, 1, 1)$ is convergent. If we put, for the sake of brevity,

$$\Phi(v_1, v_2;\, u_1, u_2) = hu_1^h u_2 v_1^{-(2h+1)}(v_1 - u_2)^{h-1}F\left(-h, -h, 1, \frac{u_2}{u_1}\frac{v_1 - u_1}{v_1 - u_2}\right)$$
$$+ u_1^{h-1}u_2(u_2 - u_1)v_1^{-2h}(v_1 - u_2)^{h-2}F',$$

where F' denotes the derivative of F with respect to the argument, we have

$$\begin{aligned}\varphi(u_1, u_2, u_3) = {} & u_1^{-h}(u_1 - u_2)^h\varphi(u_1, 0, u_3) + u_2^{-h}(u_2 - u_1)^h\varphi(0, u_2, u_3)\\ & + \int_0^{u_1}\Phi(v_1, v_2;\, u_1, u_2)\varphi(v_1, 0, u_3)dv_1\\ & + \int_0^{u_2}\Phi(v_2, v_1;\, u_2, u_1)\varphi(0, v_2, u_3)dv_2.\end{aligned} \tag{33}$$

Since the right-hand member of this equation reduces to $\varphi(0, u_2, u_3)$ for $u_1 = 0$, and since the second of equations (22) is independent of u_1, it follows that the expression (33) will satisfy the second equation, if we take for $\varphi(0, u_2, u_3)$ the solution of this second equation which for $u_2 = 0$ and $u_3 = 0$ takes the respective given values $\varphi(0, 0, u_3)$ and $\varphi(0, u_2, 0)$. In like manner the expression (33) will satisfy the third of (22), if we take for $\varphi(u_1, 0, u_3)$ that solution of this equation reducing for $u_1 = 0$ and $u_3 = 0$ to $\varphi(0, 0, u_3)$ and $\varphi(u_1, 0, 0)$ respectively.

In order to obtain these desired solutions $\varphi(0, u_2, u_3)$ and $\varphi(u_1, 0, u_3)$, we apply the Riemann method to the second and third of equations (22), and obtain

$$\begin{aligned}\varphi(0, u_2, u_3) = {} & u_2^{-h}(u_2 - u_3)^h\varphi(0, u_2, 0) + u_3^{-h}(u_3 - u_2)^h\varphi(0, 0, u_3)\\ & + \int_0^{u_2}\Phi(v_2, v_3;\, u_2, u_3)\varphi(0, v_2, 0)dv_2\\ & + \int_0^{u_3}\Phi(v_3, v_2;\, u_3, u_2)\varphi(0, 0, v_3)dv_3,\end{aligned}$$

$$\begin{aligned}\varphi(u_1, 0, u_3) = {} & u_3^{-h}(u_3 - u_1)^h\varphi(0, 0, u_3) + u_1^{-h}(u_1 - u_3)^h\varphi(u_1, 0, 0)\\ & + \int_0^{u_3}\Phi(v_3, v_1;\, u_3, u_1)\varphi(0, 0, v_3)dv_3\\ & + \int_0^{u_1}\Phi(v_1, v_3;\, u_1, u_3)\varphi(v_1, 0, 0)dv_1.\end{aligned}$$

Hence when these expressions are substituted in (33) we have the solution of (22) which is determined by the given initial values of $\varphi(u_1, 0, 0)$, $\varphi(0, u_2, 0)$ and $\varphi(0, 0, u_3)$.

PRINCETON UNIVERSITY.

ON A SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE.

BY T. H. GRONWALL.

1. It is the purpose of this note to investigate the problem of integrating the system

$$
(1)\qquad
\begin{aligned}
&\frac{\partial^2\theta}{\partial u_1\partial u_2}+\frac{\partial\theta}{\partial u_1}-\frac{\partial\theta}{\partial u_2}+3\theta=0,\\
&\frac{\partial^2\theta}{\partial u_2\partial u_3}+\frac{\partial\theta}{\partial u_2}-\frac{\partial\theta}{\partial u_3}+3\theta=0,\\
&\frac{\partial^2\theta}{\partial u_3\partial u_1}+\frac{\partial\theta}{\partial u_3}-\frac{\partial\theta}{\partial u_1}+3\theta=0,
\end{aligned}
$$

which occurs in the preceding paper.* We begin by showing, by means of the Riemann integration method, that a solution $\theta = \theta\,(u_1, u_2, u_3)$ of (1) is uniquely determined by its values on the three coördinate axes, viz., $\theta\,(u_1, 0, 0)$, $\theta\,(0, u_2, 0)$ and $\theta\,(0, 0, u_3)$.

2. To integrate an equation

$$
(2)\qquad \frac{\partial^2 z}{\partial x\partial y}+a(x,y)\frac{\partial z}{\partial x}+b(x,y)\frac{\partial z}{\partial y}+c(x,y)z=0
$$

by the Riemann method, we form the adjoint equation

$$
(3)\qquad \frac{\partial^2 t}{\partial\xi\partial\eta}-\frac{\partial}{\partial\xi}(a(\xi,\eta)t)-\frac{\partial}{\partial\eta}(b(\xi,\eta)t)+c(\xi,\eta)t=0,
$$

and determine a solution

$$t = t(\xi, \eta;\, x, y)$$

by the initial conditions

$$
(4)\qquad
\begin{aligned}
&t=e^{\int_x^\xi b(\xi,y)d\xi} &&\text{for}\quad \eta=y,\\
&t=e^{\int_y^\eta a(x,\eta)d\eta} &&\text{for}\quad \xi=x;
\end{aligned}
$$

then the solution of (2) which takes the given values $z(x, 0)$ on the x-axis and $z(0, y)$ on the y-axis is uniquely determined, and is given by

$$z(x,y)=t(x,0;\,x,y)z(x,0)+t(0,y;\,x,y)z(0,y)-t(0,0;\,x,y)z(0,0)$$

* L. P. Eisenhart, Triply conjugate systems with equal point invariants.

(5)
$$+\int_0^x\left[b(\xi,0)t(\xi,0;x,y)-\frac{\partial}{\partial\xi}t(\xi,0;x,y)\right]z(\xi,0)d\xi$$
$$+\int_0^y\left[a(0,\eta)t(0,\eta;x,y)-\frac{\partial}{\partial\eta}t(0,\eta;x,y)\right]z(0,\eta)d\eta.$$

3. We now apply this to the first of equations (1). The adjoint equation is

$$\frac{\partial^2 t}{\partial v_1\partial v_2}-\frac{\partial t}{\partial v_1}+\frac{\partial t}{\partial v_2}+3t=0$$

and the conditions (4) become

$$t=e^{\int_{u_1}^{v_1}-dv_1}=e^{u_1-v_1}\qquad\text{for}\qquad v_2=u_2,$$

$$t=e^{\int_{u_2}^{v_2}dv_2}=e^{v_2-u_2}\qquad\text{for}\qquad v_1=u_1.$$

Writing $t=e^{u_1-u_2}\cdot e^{v_2-v_1}\varphi$, the adjoint equation becomes

$$\frac{\partial^2\varphi}{\partial v_1\partial v_2}+4\varphi=0,$$

with the initial conditions $\varphi=1$ for $v_1=u_1$ and for $v_2=u_2$. With the notation

(6)
$$f_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}x^n}{n!(n+\nu)!}=\frac{1}{(2\sqrt{x})^\nu}J_\nu(4\sqrt{x}),$$

where J_ν is the Bessel function of order ν, it is readily shown that

(7)
$$\varphi(v_1,v_2;u_1,u_2)=f_0((u_1-v_1)(u_2-v_2))$$

satisfies both the adjoint equation and the initial conditions. From (6) and (7) it follows that

$$\frac{\partial}{\partial v_1}\varphi(v_1,0;u_1,u_2)=4u_2f_1(u_2(u_1-v_1)),$$

$$\frac{\partial}{\partial v_2}\varphi(0,v_2;u_1,u_2)=4u_1f_1(u_1(u_2-v_2)),$$

and (5) gives

(8)
$$\theta(u_1,u_2,u_3)=e^{-u_2}\theta(u_1,0,u_3)+e^{u_1}\theta(0,u_2,u_3)-e^{u_1-u_2}f_0(u_1u_2)\theta(0,0,u_3)$$
$$-e^{-u_2}\int_0^{u_1}e^{u_1-v_1}\cdot 4u_2f_1(u_2(u_1-v_1))\theta(v_1,0,u_3)dv_1$$
$$-e^{u_1}\int_0^{u_2}e^{v_2-u_2}\cdot 4u_1f_1(u_1(u_2-v_2))\theta(0,v_2,u_3)dv_2.$$

4. Since the coefficients in (1) are constant, it follows that $\theta(0, u_2, u_3)$ must satisfy the second, and $\theta(u_1, 0, u_3)$ the third of these equations. The application of the Riemann method gives, in the same way as before,

$$
\begin{aligned}
\theta(0, u_2, u_3) = {} & e^{-u_3}\theta(0, u_2, 0) + e^{u_2}\theta(0, 0, u_3) - e^{u_2-u_3}f_0(u_2u_3)\theta(0, 0, 0) \\
& - e^{-u_3}\int_0^{u_2} e^{u_2-v_2}\cdot 4u_3f_1(u_3(u_2 - v_2))\theta(0, v_2, 0)dv_2 \\
& - e^{u_2}\int_0^{u_3} e^{v_3-u_3}\cdot 4u_2f_1(u_2(u_3 - v_3))\theta(0, 0, v_3)dv_3
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\theta(u_1, 0, u_3) = {} & e^{-u_1}\theta(0, 0, u_3) + e^{u_3}\theta(u_1, 0, 0) - e^{u_3-u_1}f_0(u_3u_1)\theta(0, 0, 0) \\
& - e^{-u_1}\int_0^{u_3} e^{u_3-v_3}\cdot 4u_1f_1(u_1(u_3 - v_3))\theta(0, 0, v_3)dv_3 \\
& - e^{u_3}\int_0^{u_1} e^{v_1-u_1}\cdot 4u_3f_1(u_3(u_1 - v_1))\theta(v_1, 0, 0)dv_1.
\end{aligned} \tag{10}
$$

Substituting (9) and (10) in (8), we see that $\theta(u_1, u_2, u_3)$ is uniquely determined when $\theta(u_1, 0, 0)$, $\theta(0, u_2, 0)$ and $\theta(0, 0, u_3)$ are given. We may simplify this solution formally by remarking that, since $\theta(u_1, u_2, u_3)$ is a linear functional of the initial values, it may be written as the sum of four solutions

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3, \tag{11}$$

where θ_0 is any particular solution such that $\theta_0(0, 0, 0) = \theta(0, 0, 0)$, and the initial values of θ_1, θ_2 and θ_3 are given by the following table

	θ_1	θ_2	θ_3
on u_1-axis	$\theta(u_1, 0, 0) - \theta_0(u_1, 0, 0)$	0	0
" u_2- "	0	$\theta(0, u_2, 0) - \theta_0(0, u_2, 0)$	0
" u_3- "	0	0	$\theta(0, 0, u_3) - \theta_0(0, 0, u_3)$.

5. A solution θ_0 is readily found by writing

$$\theta_0 = \theta(0, 0, 0)e^{\lambda_1u_1 + \lambda_2u_2 + \lambda_3u_3} \tag{12}$$

and substituting in (1), we find the following conditions for the constants $\lambda_1, \lambda_2, \lambda_3$:

$$
\begin{aligned}
\lambda_1\lambda_2 + \lambda_1 - \lambda_2 + 3 &= 0, \\
\lambda_2\lambda_3 + \lambda_2 - \lambda_3 + 3 &= 0, \\
\lambda_3\lambda_1 + \lambda_3 - \lambda_1 + 3 &= 0.
\end{aligned}
$$

The general solution of these equations is

$$\lambda_1 = \frac{1}{t}, \qquad \lambda_2 = \frac{1 + 3t}{t - 1}, \qquad \lambda_3 = \frac{1 - 3t}{t + 1}, \tag{13}$$

where t is arbitrary. For instance, making $t = \pm i\sqrt{3}$ and adding the two resulting expressions (12), we may write

$$\theta_0 = \theta(0, 0, 0) \cos \sqrt{3}(u_1 + u_2 + u_3).$$

6. Writing $\theta(u_1, 0, 0) - \theta_0(u_1, 0, 0) = h(u_1)$ so that $h(0) = 0$, the solution $\theta_1(u_1, u_2, u_3)$ has the values $h(u_1)$, 0, 0 on the three coördinate axes. Equation (9) gives $\theta_1 (0, u_2, u_3) = 0$, (10) becomes

$$\theta_1(u_1, 0, u_3) = e^{u_3}h(u_1) - e^{u_3}\int_0^{u_1} e^{v_1-u_1}\cdot 4u_3 f_1(u_3(u_1 - v_1)h(v_1)dv_1,$$

and substituting in (8), we find

$$\begin{aligned}\theta_1(u_1, u_2, u_3) = e^{u_3-u_2}\Big\{ h(u_1) &- 4u_3\int_0^{u_1} e^{v_1-u_1}f_1(u_3(u_1 - v_1))h(v_1)dv_1 \\ &- 4u_2\int_0^{u_1} e^{u_1-v_1}f_1(u_2(u_1 - v_1))h(v_1)dv_1 \\ &+ 16u_2u_3\int_0^{u_1} e^{u_1-w_1}f_1(u_2(u_1-w_1))dw_1\int_0^{w_1} e^{v_1-u_1}f_1(u_3(w_1-v_1))h(v_1)dv_1\Big\}.\end{aligned}$$

Changing the order of integration in the repeated integral, this becomes

$$\theta_1(u_1, u_2, u_3) = e^{u_3-u_2}\left[h(u_1) + \int_0^{u_1} \Phi(u_1 - v_1, u_2, u_3)h(v_1)dv_1\right] \tag{14}$$

where

$$\begin{aligned}\Phi(u_1 - v_1, u_2, u_3) = 16u_2u_3 &\int_{v_1}^{u_1} e^{v_1-w_1}f_1(u_3(w_1 - v_1))f_1(u_2(u_1 - w_1))dw_1 \\ &- 4u_2e^{u_1-v_1}f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1}f_1(u_3(u_1 - v_1))\end{aligned}$$

or, making $w_1 = v_1 + (u_1 - v_1)x$,

$$\begin{aligned}\Phi(u_1 - v_1, u_2, u_3) &= 16u_2u_3(u_1 - v_1) \\ &\times\int_0^1 e^{-(u_1-v_1)x}f_1(u_3(u_1 - v_1)x)f_1(u_2(u_1 - v_1)(1 - x))dx \\ &- 4u_2e^{u_1-v_1}f_1(u_2(u_1 - v_1)) - 4u_3e^{v_1-u_1}f_1(u_3(u_1 - v_1)).\end{aligned} \tag{15}$$

The corresponding expressions for θ_2 and θ_3 are found by cyclic permutation of the variables.

7. In some cases, $\theta_1(u_1, u_2, u_3)$ may be expressed as a contour integral. From (12) and (13) it follows that a solution of (1) is given by

$$\theta_1(u_1, u_2, u_3) = \frac{1}{2\pi i}\int_C e^{\frac{u_1}{t}+\frac{1+3t}{t-1}u_2+\frac{1-3t}{t+1}u_3}f(t)dt, \tag{16}$$

where the contour C encloses $t = 0$, but neither $t = 1$ nor $t = -1$, and $f(t)$

is holomorphic inside C. For $u_1 = 0$, the integrand is holomorphic inside C, so that $\theta_1(0, u_2, u_3) = 0$ and in particular $\theta_1(0, u_2, 0) = \theta_1(0, 0, u_3) = 0$. For $u_2 = u_3 = 0$, we find

$$h(u_1) = \theta_1(u_1, 0, 0) = \frac{1}{2\pi i}\int_C e^{u_1 t} f(t)dt.$$

Let $f(t) = \sum_0^\infty c_\nu t^\nu$ converge for $|t| < r$ and deform C into the circle $|t| = \rho < r$; then

$$h(u_1) = \frac{1}{2\pi i}\int_{|t|=\rho} \sum_{\mu,\,\nu=0}^{\infty} \frac{c_\nu u_1^\mu}{\mu!} t^{\nu-\mu} dt = \sum_{\nu=0}^{\infty} \frac{c_\nu u_1^{\nu+1}}{(\nu+1)!}.$$

When

$$h(u_1) = \sum_{\nu=1}^{\infty} a_\nu u_1^\nu$$

is given, it follows that $c_\nu = (\nu + 1)!a_{\nu+1}$ and

$$f(t) = \sum_{\nu=0}^{\infty} (\nu + 1)!a_{\nu+1} t^\nu, \tag{17}$$

provided that this series has a radius of convergence greater than zero. For instance, when $h(u_1) = e^{u_1} - 1$, then $f(t) = 1/(1 - t)$.

SOME PROPERTIES OF CIRCLES AND RELATED CONICS.

By James H. Weaver.

Let there be given two circles C_1 and C_2 with centers O_1 and O_2 respectively. The locus of the center of a circle tangent to C_1 and C_2 will be two confocal conics, E_1 and H_1, with foci O_1 and O_2. Conversely, with every two confocal conics there may be associated two circles having the common foci as centers, and with radii r_1 and r_2, such that $r_1 + r_2 = 2a$ and $r_2 - r_1 = 2a'$, where a and a' denote the semi-major axes of the two confocals.

Theorem I: If from any point P on the radical axis of the two circles C_1 and C_2, tangents be drawn to C_1, C_2, E_1 and H_1, the points of contact lie on four straight lines, two of which pass through O_1 and two through O_2.

Proof: Let the points of contact of the tangents to C_1 and C_2 be B, B', A, A' and to E_1 be D and D' (See Figure).

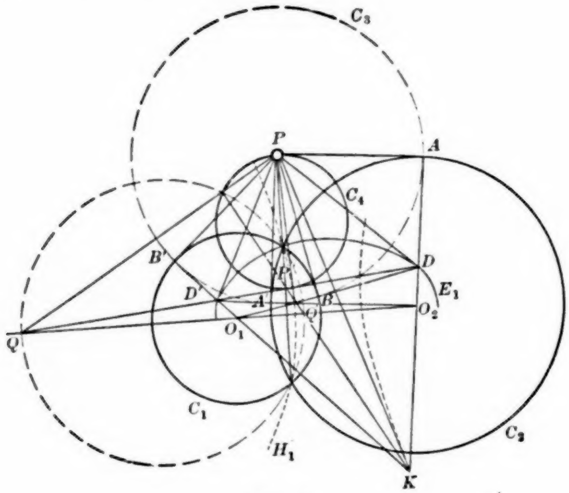

Fig. 1.

Since P is on the radical axis of C_1 and C_2, $PA = PB = PA' = PB'$, and the four points A, B, A' and B' lie on a circle C_3 with center P, and which cuts C_1 and C_2 orthogonally. Therefore O_1B and O_2A are tangent to C_3. Let O_1B and O_2A intersect in D_1'. Then D_1' is on the conic E_1 because it is the center of a circle tangent to C_1 and C_2. But PD_1' bisects

the angle formed by O_1B and O_2A. It is therefore tangent to E_1, and $D_1' = D$. Therefore D and A are collinear with O_2 and D and B with O_1. Similarly it may be shown that all the points of contact of the tangents from P lie on the lines O_2A, O_2A', O_1B and O_1B'.

The two following theorems follow immediately.

Theorem II: The four points of contact of the tangents from P to E_1 and H_1 determine a quadrangle the vertices of whose diagonal triangle are the centers of C_1 and C_2 and P' the conjugate of P with respect to the system.*

Theorem III: The four lines on which the points of contact lie determine a quadrilateral, the sides of whose diagonal triangle are the line of centers of C_1 and C_2 and the polars of P with respect to E_1 and H_1.

Since this triangle is self polar with respect to C_3, P is its ortho center. The two following theorems may be easily proved analytically.

Theorem IV: The polars of P with respect to E_1 and H_1 pass through the centers of perspective of C_1 and C_2.

Theorem V: The perpendiculars from P to the polars of E_1 and H_1 intersect these polars on the circle of similitude of C_1 and C_2.

Moreover a circle C_4 on PP' as diameter will pass through this point of intersection. We therefore have

Theorem VI: The polars of P with respect to E_1 and H_1 intersect the polars of P' with respect to H_1 and E_1 respectively on the circle with PP' as diameter.

* The polars of P with respect to C_1, C_2, E_1 and H_1 intersect in P' on the radical axis of C_1 and C_2. This may be easily proven analytically.

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INTEGRALS IN AN INFINITE NUMBER OF DIMENSIONS.

By P. J. Daniell.

1. In a recent issue of the Annals of Mathematics* there appeared a paper by the author on "A General Form of Integral." In it a method was given whereby integrals could be defined for functions of general elements (p) which could theoretically be of any character. The author was then unable to give a definite example in which the elements (p) were points in a denumerably infinite number of dimensions. According to Hildebrandt,† a definite example, which does not reduce to an infinite series or to an integral over a finite number of dimensions, is still lacking. Even Fréchet‡ did not give any example which was sufficiently general. Since the publication of his previous paper, the author has found two examples, and this is now an account of them. The first is a generalization of the Lebesgue integral in the interval $(0 \leq x \leq 1)$; the second an infinitely multiple Stieltjes integral of positive type. The author hopes to publish soon a still wider generalization of the Stieltjes integral.

For the purpose before us it is necessary to use certain properties of points and functions in a denumerable number of dimensions. Some of these properties are obtained by Fréchet§ in his thesis to which reference will be made by means of notation F., p. 40, for example, referring to page 40.

The elements (p) are points in a denumerable number of dimensions, that is to say, having a denumerable number of coördinates,

$$p = (x_1, x_2, \cdots x_n, \cdots).$$

Fréchet (F., p. 40) defines the "écart" of two points p, p', to be

$$(p, p') = \frac{|x_1 - x_1'|}{1 + |x_1 - x_1'|} + \cdots + \frac{1}{n!}\frac{|x_n - x_n'|}{1 + |x_n - x_n'|} + \cdots,$$

and the class of points, for which this écart is defined, he calls (E_ω).

By F., p. 45, a function $f(p)$ of elements p of class (E_ω) is said to be *continuous*, if $\lim f(p_r) = f(p)$, as the sequence $\{p_r\}$ approaches p.

Daniell's theory was based on a class of functions, T_0, satisfying the following requirements: Each function must be bounded, and the class

* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), p. 279.

† T. H. Hildebrandt, Bulletin of the American Mathematical Society, vol. 24 (1917), p. 116.

‡ M. Fréchet, Bulletin de la Société de France, vol. 43 (1915), p. 249.

§ M. Fréchet, Rendiconti di Circolo matematico di Palermo, vol. 22 (1906), pp. 1–74.

T_0 is closed with respect to the operations (op. C) multiplication by a constant, (op. A) addition of two functions, and "logical addition and multiplication" of two functions. It simplifies the theory, if we notice that when T_0 is closed with respect to (op. C) (op. A), then closure with respect to "logical addition and multiplication" is equivalent logically to closure with respect to (op. M), the operation of taking the modulus. For, on the one hand,

$$|f| = f \vee 0 - f \wedge 0,$$

and $0 = 0 \times f$ belongs to T_0; on the other hand,

$$f \vee g = \tfrac{1}{2}[f + g + |f - g|],$$

$$f \wedge g = \tfrac{1}{2}[f + g - |f - g|].$$

We shall choose the class T_0 to be the class of functions

$$f(p) = f(x_i, x_j, \cdots x_r),$$

which are functions of the finite number of variables, $x_i, x_j, \cdots x_r$, (chosen from the variables, $x_1, x_2, \cdots x_n, \cdots$) and bounded and continuous in the domain considered, that is to say, in the first case, in the finite domain $(0 \leq x_i \leq 1, \cdots 0 \leq x_r \leq 1)$, and in the second case, for all finite values of $x_i, x_j, \cdots, x_r$. If f and g are two such functions, e. g., $f(x_i, \cdots, x_r)$, $g(x_p, \cdots, x_t)$, their sum will be a function of $x_i, x_j, \cdots x_r, x_p, \cdots x_t$ (where some of these variables may be identical) and $f + g$ will be also bounded and continuous. The class T_0 will evidently satisfy all the required closure conditions.

2. **Simple Integral.** For any function

$$f(p) = f(x_i, x_j, \cdots, x_r)$$

of class T_0 we define

$$I(f) = \int_0^1 \cdots \int_0^1 f(x_i, \cdots x_r) dx_i dx_j \cdots dx_r.$$

This definition is possible since f is a continuous function of a finite number of variables. Then $I(f)$ is finite and satisfies the conditions,

(C) $\qquad I(cf) = cI(f)$, if c is any constant,

(A) $\qquad I(f_1 + f_2) = I(f_1) + I(f_2)$,

(P) $\qquad I(f) \geq 0$, if $f(p) \geq 0$ for all p.

We need to give an extended proof for (L) only, namely,

(L) $\qquad$ If $f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n$ for every p,

then

$$\lim I(f_n) = 0.$$

In the first place,

$$|I(f)| \leq \max_p |f(p)|.$$

In this case $I(f_n) \leq \max f_n(p)$, or

$$\lim_{n=\infty} I(f_n) \leq \lim_{n=\infty} \max_p f_n(p).$$

The condition (L) will be satisfied if $\max f_n(p)$ approaches 0 with $1/n$. Assume that it does not, then we can find a $k > 0$, such that $\max f_n(p) \geq k$ for all n. But $f_n(p)$ is continuous and therefore attains its maximum at least once (F., p. 45), or the set E_n of points such that $f_n(p) \geq k$ contains at least one point. Moreover, since $f_{n+1}(p) \leq f_n(p)$ for all p, the set E_{n+1} is contained in the set E_n. The sets E_n are closed, for if $\{p_r\}$ $(r = 1, 2, \cdots)$ is a sequence of points contained in E_n, approaching p as a limit, then

$$f_n(p_r) \geq k, \qquad r = 1, 2, \cdots,$$

$$f_n(p) = \lim f_n(p_r) \geq k,$$

or p belongs to E_n.

The set $(0 \leq x_n \leq 1)(n = 1, 2, \cdots)$ is compact by F., p. 42, and it follows by F., p. 7, that there is at least one point common to every E_n, that is to say, there is a point p, such that $f_n(p) \geq k$ for all n. This is contrary to the hypothesis, that $\lim f_n(p) = 0$ for all p. Then our assumption must be incorrect, or

$$\lim_n \max_p f_n(p) = 0,$$

and thus (L) is proved. Our integral satisfies all the required conditions for functions of class T_0. We can then extend it to all summable functions according to the methods laid down by Daniell (l. c.). In particular, any function which can be obtained from functions of class T_0 by successive operations such as multiplication by constants, addition, forming the modulus and taking the limit of a sequence of functions bounded in their set, will be summable. For if K is any constant, $\varphi(p) = K$ is a member of class T_0, and summable; and we make use of Daniell, theorems (7·2, 7·3, 7·4, 7·7).

For example *any continuous function is summable.* For (1) it is bounded (F., p. 45) or a finite K can be found such that $|f(p)| \leq K$ for all p; (2) it is the limit of a sequence of functions of class T_0, bounded in their set.

If $p = (x_1, x_2, \cdots x_n, x_{n+1}, \cdots)$, consider the function

$$f_n(p) = f(x_1, x_2, \cdots x_n, 0, 0, \cdots).$$

This function is of class T_0 and $|f_n(p)| \leq K$, for all n and p.

If

$$p_n = (x_1, x_2, \cdots x_n, 0, 0, \cdots),$$

$$f_n(p) = f(p_n) \text{ and the "écart"}$$

$$(p, p_n) = \frac{1}{(n+1)!} \frac{|x_{n+1}|}{1 + |x_{n+1}|} + \cdots$$

approaches the limit 0 with $1/n$. Or, since $f(p)$ is continuous,

$$f(p) = \lim f(p_n) = \lim f_n(p).$$

3. **Measure of a Set.** Corresponding to any set of points, E, contained in the interval E_0, $(0 \leq x_n \leq 1)(n = 1, 2, \cdots)$, we can define a function equal to 1 on E, and equal to 0 otherwise. We define the measure of a set as the integral of this corresponding function (if it is summable). This measure will be non-negative, additive, and bounded. It is convenient to introduce a symbol "l" to denote either of the inequalities $<$, $\leqq$, and in what follows, it does not necessarily denote the same throughout. The function corresponding to the interval, $a_1 l x_1 l b_1, \cdots, a_n l x_n l b_n$, $0 \leq x_{n+1} \leq 1$, $0 \leq x_{n+2} \leq 1, \cdots$ (where $0 \leq a_1 \leq b_1 \leq 1, \cdots 0 \leq a_n \leq b_n \leq 1$), is a limit of functions, bounded in their set, belonging to class T_0, and the measure of this interval is

$$(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

If we are given a sequence of such intervals, the measure of the limiting interval

$$a_1 l x_1 l b_1, \cdots a_n l x_n l b_n, a_{n+1} l x_{n+1} l b_{n+1}, \cdots,$$

is the limit of the infinite product,

$$(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n) \cdots.$$

Let $u_n = 1 - b_n + a_n$, then $0 \leq u_n \leq 1$ and the product becomes

$$(1 - u_1)(1 - u_2) \cdots (1 - u_n) \cdots.$$

This infinite product converges if the series of non-negative terms $u_1 + u_2 + \cdots + u_n + \cdots$ converges, and diverges to 0, if the series is divergent. In either case the interval is measurable.

For example, the interval

$$A \ (0 \leqq x_n < 1)(n = 1, 2, \cdots),$$

is measurable and has the measure 1, while the interval

$$E_\epsilon \ (0 \leqq x_n \leqq 1 - \epsilon)(n = 1, 2, \cdots)(\epsilon > 0),$$

has the measure 0 however small the positive ϵ may be. It would seem

as if the intervals E_ϵ approach A as ϵ approaches 0, leading to a contradiction, but this is not the case. Consider the point

$$p = (\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \cdots).$$

This point lies in A, but no $\epsilon > 0$ can be found such that p lies in E_ϵ. In fact, A is rather the outer limiting set of all sets of the type

$$(0 \leqq x_n \leqq 1 - \epsilon_n)(n = 1, 2, \cdots),$$

and we can make the measure of this interval approach 1 as nearly as we please, by making the series $\Sigma\epsilon_n$ convergent and its sum sufficiently small.

Even in a triple integral, it is difficult to find examples, in which the integration can be performed analytically, if we disregard cases which reduce to simple integrals immediately. It is yet more difficult for an infinitely multiple integral. The following example is unsatisfactory because it is obtained by an infinite series of simple integrals. Let A be a point contained in E_0,

$$A = (a_1, a_2, \cdots, a_n, \cdots),$$

then consider

$$f(p) = \text{écart}\ (p, A) = \frac{|x_1 - a_1|}{1 + |x_1 - a_1|} + \cdots + \frac{1}{n!}\frac{|x_n - a_n|}{1 + |x_n - a_n|} + \cdots.$$

$I(f)$ will be

$$e - 1 - \sum_{n=1}^{\infty} \frac{1}{n!} \log (1 + a_n)(2 - a_n).$$

If $A = (0, 0, \cdots)$ or $(1, 1, 1, \cdots)$,

$$I(f) = (e - 1)(1 - \log 2).$$

4. **Multiple Stieltjes integral of positive type.** Let $\beta_1(t), \beta_2(t), \cdots$ be any denumerable set of functions of t, defined and nondecreasing from $t = -\infty$ to $t = +\infty$, and such that

$$\beta_n(-\infty) = \lim_{t \doteq -\infty} \beta_n(t) = 0,$$

$$\beta_n(+\infty) = \lim_{t \doteq +\infty} \beta_n(t) = 1.$$

Lemma 1. Let $B_n(t)(t > 0)$ denote

$$\int_{-t}^{+t} d\beta_n(t) = \beta_n(t) - \beta_n(-t) \geqq 0.$$

Given any $\epsilon_n > 0$, we can find M_n so that

$$B_n(M_n) > 1 - \epsilon_n.$$

For $B_n(t) = 1 - [1 - \beta_n(t)] - \beta_n(-t)$, and

$$\lim_{t \doteq \infty} [1 - \beta_n(t)] = 0, \qquad \lim_{t \doteq \infty} \beta_n(-t) = 0.$$

Lemma 2. Denote $B_n(M_n)$ by B_n, then $0 \leq B_n \leq 1$, and if $M_i \leqq M_i'$,

$$M_j \leqq M_j', \cdots, M_r \leqq M_r'.$$

$$B_i'B_j' \cdots B_r' - B_iB_j \cdots B_r \leqq (B_i' - B_i) + (B_j' - B_j) + \cdots + (B_r' - B_r).$$

For

$$\begin{aligned} B_i'B_j' \cdots B_r' - B_iB_j' \cdots B_r' &= (B_i' - B_i)B_j' \cdots B_r' \\ &\leqq B_i' - B_i, \\ B_iB_j' \cdots B_r' - B_iB_jB_k' \cdots B_r' &= (B_j' - B_j)B_iB_k' \cdots B_r' \\ &\leqq B_j' - B_j, \end{aligned}$$

and so on.

5. **Definition of integral.** As before we choose the class T_0 to be the class of functions of a finite number of variables $(x_i, \cdots, x_r)$, bounded and continuous for all finite values of these variables. We define

$$I(f) = \int_{-\infty}^{+\infty} \cdots \int f(x_i, x_j, \cdots, x_r) d\beta(x_i) \cdots d\beta(x_r).$$

This definition is possible since f is continuous and bounded, and

$$\beta(x_i, x_j, \cdots, x_r) = \beta(x_i)\beta(x_j) \cdots \beta(x_r)$$

is a limited function of positive type, i. e., such that

$$\Delta_i\Delta_j \cdots \Delta_r\beta(x_i, \cdots, x_r) \geq 0.$$

To justify the infinite limits we denote

$$\int_{-M_i}^{+M_i}\int_{-M_j}^{+M_j} \cdots \int_{-M_r}^{+M_r} fd\beta(x_i) \cdots d\beta(x_r) = I(f;\ M_i, \cdots M_r).$$

Then, if $M_i' \geqq M_i, \cdots M_r' \geqq M_r$,

$$\begin{aligned} &| I(f;\ M_i', M_j', \cdots M_r') - I(f;\ M_i, M_j, \cdots M_r) | \\ &\qquad \leqq \max |f| \cdot [B_i'B_j' \cdots B_r' - B_iB_j \cdots B_r] \\ &\qquad \leqq \max |f| \cdot [(B_i' - B_i) + \cdots + (B_r' - B_r)], \end{aligned}$$

by Lemma 2. Then by Lemma 1, given any $\epsilon > 0$, we can find $M_i, M_j, \cdots, M_r$ so that for all $M_i' \geqq M_i, \cdots, M_r' \leqq M_r$, the last expression in a bracket is less than ϵ. The integral, so defined, satisfies the conditions $(C)(A)(P)$ and

$$| I(f) | \leqq \max |f(p)|.$$

To prove that condition (L) is also satisfied, we know, in the first place, that given any set of positive numbers $M_1, M_2, \cdots M_n, \cdots$, the domain

$$|x_n| \leq M_n, \qquad (n = 1, 2, \cdots)$$

is a finite domain (F., p. 42) and is therefore compact. By the same method of reasoning as we employed before, we can prove that given any $\epsilon > 0$ and any set of positive numbers $M_1, M_2, \cdots, M_n, \cdots$, we can find q_0, so that

$$I(f_q;\ M_1, \cdots) < \tfrac{1}{2}\epsilon \qquad (q \geqq q_0).$$

Choose the numbers $M_1, M_2, \cdots$, using lemma 1, so that,

$$B_n(M_n) > 1 - 3^{-n}\eta, \qquad (n = 1, 2, \cdots).$$

Then if f_q is a function of class T_0, and $f_q \geqq 0$,

$$\begin{aligned} I(f_q) - I(f_q;\ M_1, M_2, \cdots) &< \max_p f_q(p)\cdot\eta(3^{-1} + 3^{-2} + \cdots) \\ &< \tfrac{1}{2}\eta \max_p f_q(p), \\ &< \tfrac{1}{2}\eta \max_p f_1(p)\cdot(q = 1, 2, 3, \cdots). \end{aligned}$$

Choose $\eta = \epsilon \div \max f_1(p)$, then we can choose $M_1, M_2, \cdots$, so that *for all* q

$$I(f_q) - I(f_q;\ M_1, M_2, \cdots) < \tfrac{1}{2}\epsilon,$$

and then with this choice made we can find q_0 so that

$$I(f_q;\ M_1, M_2, \cdots) < \tfrac{1}{2}\epsilon \qquad (q \geqq q_0).$$

Then combining these, given any $\epsilon > 0$ we can find q_0, so that

$$I(f_q) < \epsilon \quad (q \geqq q_0),$$

or limit $I(f_n) = 0$.

As before, we can extend the definition to all continuous, bounded functions and to all functions obtained by a succession of operations of addition, multiplication by a constant, taking the modulus, and taking the limit of a sequence of functions bounded in their set.

Example. Let

$$\beta_n(t) = \int_{-\infty}^{t} e^{-\pi t^2}dt, \qquad (n = 1, 2, \cdots);$$

then

$$\beta_n(\infty) = \int_{-\infty}^{+\infty} e^{-\pi t^2}dt = 1.$$

If $f(p)$ is any bounded continuous function in E_ω, the integral

$$I(f) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdots f(x,_1 x_2, \cdots x_n, \cdots) e^{-\pi \sum_1^\infty x_n^2} dx_1 \cdots dx_n \cdots$$

can be defined, and we may add the convention that when Σx_n^2 is divergent,

$$e^{-\pi \sum_1^\infty x_n^2} = 0.$$

If

$$\begin{aligned} \beta_n(t) &= 0, \qquad t < 0, \\ &= t, \qquad 0 \leq t \leq 1, \qquad (n = 1, 2, \cdots) \\ &= 1, \qquad t > 1, \end{aligned}$$

then this more general integral reduces to the case first considered.

Note. In finding an example of the simple integral in an infinite number of dimensions, the author desired to invent one which could readily be evaluated. For this reason it was somewhat unsatisfactory. But if we wish a genuine example and do not require its evaluation we may set up the integral,

$$I(f) = \int_0^1 \cdots \int_0^1 \cdots f(p) dx_1 dx_2 \cdots dx_n \cdots,$$

where

$$f(p) = [e - (p, A)]^{1/2}$$

(p, A) = écart between p and a fixed point A.

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ON THE DIFFERENTIABILITY OF THE SOLUTION OF A DIFFERENTIAL EQUATION WITH RESPECT TO A PARAMETER.

By J. F. Ritt.

There will be given in this note a proof of the differentiability of the solution of a single differential equation, of the first order, with respect to a parameter, which does not employ the mechanism of any existence proof. All other treatments of this problem which the writer has seen consist in studying the behavior of the derivatives with respect to the parameter, of the successive approximations to the solution which the existence proof produces. The method usually followed in treatises on analysis is based on the Cauchy-Lipschitz process. Bliss* has recently given a very general treatment of this problem, using the Picard existence proof. Moulton† has also given a proof which is based on the Picard process.

The method given here can also be used in the proof of the differentiability of the solution of a single differential equation with respect to the constant of integration. It does not seem possible to give a simple generalization of the method for systems of differential equations.

Let the equation be

$$\frac{dy}{dx} = f(x, y, a), \tag{1}$$

where $f(x, y, a)$, $f_y(x, y, a)$ and $f_a(x, y, a)$ exist and possess conjoint continuity with respect to all three variables for

$$x_0 \leqq x \leqq x_0 + h, \qquad y_0 - k \leqq y \leqq y_0 + k, \qquad \alpha_1 \leqq a \leqq \alpha_2.$$

We assume also that for every a in the interval (α_1, α_2) there exists a function $y(x)$, defined on the interval $(x_0, x_0 + h)$, satisfying equation (1) in this interval, assuming only values which lie between $y_0 - k$ and $y_0 + k$, and taking on the value y_0 for $x = x_0$. The continuity of $f_y(x, y, a)$ implies the satisfaction of the Lipschitz condition. Hence there will exist only a single solution of (1) for a given value of a, which reduces to y_0 for $x = x_0$.

We shall show that this solution possesses a derivative y_a with respect to a, which satisfies the equation

$$\frac{d}{dx} y_a = f_y y_a + f_a. \tag{2}$$

* Bulletin of the American Mathematical Society, Oct., 1918.

† Major Moulton's paper was issued in photostat form by the Technical Staff of the Ordnance Department.

Let y_1 be that solution of the equation for the value a_1 of the parameter, which reduces to y_0 for $x = x_0$. We have

$$\frac{d}{dx}(y_1 - y) = f(x, y_1, a_1) - f(x, y, a). \tag{3}$$

Hence if $a_1 \neq a$, and $y_1 \neq y$, we have

$$\frac{d}{dx}\left(\frac{y_1 - y}{a_1 - a}\right) = \frac{f(x, y_1, a_1) - f(x, y, a_1)}{y_1 - y}\,\frac{y_1 - y}{a_1 - a} + \frac{f(x, y, a_1) - f(x, y, a)}{a_1 - a}. \tag{4}$$

Let

$$U = \frac{f(x, y_1, a_1) - f(x, y, a_1)}{y_1 - y}, \qquad V = \frac{f(x, y, a_1) - f(x, y, a)}{a_1 - a}$$

respectively, for $y_1 \neq y$ and for $a_1 \neq a$, and

$$U = f_y(x, y, a_1), \qquad V = f_a(x, y, a),$$

respectively, for $y_1 = y$ and for $a_1 = a$.

Then

$$\frac{d}{dx}\left(\frac{y_1 - y}{a_1 - a}\right) = U\frac{y_1 - y}{a_1 - a} + V, \tag{5}$$

even when $y_1 = y$.

We have

$$U = f_y[x, y + \theta_1(y_1 - y), a_1], \qquad V = f_a[x, y, a + \theta_2(a_1 - a)], \tag{6}$$

where θ_1 and θ_2 lie between 0 and 1.

The function U is conjointly continuous in all of its arguments. This is evident from the definition of U, except in the case of $y_1 = y$. For that case, the continuity follows immediately from the first of equations (6). Similarly, V is continuous in all of its arguments.

We have from equation (5), since $y_1 - y = 0$, for $x = x_0$,

$$\frac{y_1 - y}{a_1 - a} = e^{\int_{x_0}^{x} U dx} \int_{x_0}^{x} e^{\int_{x_0}^{x} -U dx}\, V dx. \tag{7}$$

In this last equation, U and V can be regarded of functions of x, a and a_1, the functions y and y_1 being known. Since the functions U and V are bounded, the ratio of $y_1 - y$ to $a_1 - a$ is also a bounded function of x, a and a_1. Hence, as a_1 approaches a, y_1 approaches y uniformly with respect to x. We see from equations (6) that U and V approach, uniformly with respect to x, the functions $f_y(x, y, a)$ and $f_a(x, y, a)$ respectively. When a continuous function approaches a limit uniformly, the indefinite integral of the function approaches uniformly the indefinite integral of the limit.

Thus the first member of equation (7) approaches a limit y_a, and we have

$$y_a = e^{\int_{x_0}^{x} f_y dx} \int_{x_0}^{x} e^{-\int_{x_0}^{x} f_y dx} f_a dx.$$

Equation (2), which was to be obtained, is found by differentiating this last result.

The method employed above can be used to prove that the solution y is differentiable with respect to its initial value y_0. We shall not go into the details.

WASHINGTON, D. C.,
Apr 1, 1919.

NOTE ON THE DERIVATIVES WITH RESPECT TO A PARAMETER OF THE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS.

By T. H. Gronwall.

1. For a single equation, Dr. Ritt has solved the problem indicated in the title by a very simple and direct method which presupposes only that the solution of the equation has been proved to exist, but makes no use of the properties of the particular type of approximations on which this existence proof is based. Being founded on the integrability of a linear differential equation of the first order by quadratures, Dr. Ritt's proof cannot be extended immediately to a system of equations. It is the purpose of the present note to show how this difficulty may be overcome; similar proofs of the differentiability of the solution with respect to the initial values of both dependent and independent variables are also given.

2. In the following, the subscripts λ, μ, ν will take the values $1, 2, \cdots, n$. Let a be a parameter, and consider the system of differential equations

$$\frac{dy_\nu}{dx} = f_\nu(x; y_1, \cdots, y_n; a) \tag{1}$$

with the initial conditions

$$y_\nu = y_\nu^0 \qquad \text{for} \qquad x = x_0; \tag{2}$$

we assume that all the functions

$$f_\nu, f_{\mu\nu} = \frac{\partial f_\nu}{\partial y_\mu} \qquad \text{and} \qquad f_{0\nu} = \frac{\partial f_\nu}{\partial a}$$

are continuous for

$$x_0 \leqq x \leqq x_0 + h, \qquad y_\nu^0 - k_\nu \leqq y_\nu \leqq y_\nu^0 + k_\nu, \qquad a_1 \leqq a \leqq a_2. \tag{3}$$

We also assume that it has already been proved that the equations (1) possess a unique system of solutions with the given initial conditions, defined on the interval $(x_0, x_0 + h)$ and each y_ν assuming, for $x_0 \leqq x \leqq x_0 + h$ and every a in (a_1, a_2), only values between $y_\nu^0 - k_\nu$ and $y_\nu^0 + k_\nu$. Then *the partial derivatives $\partial y_\nu/\partial a$ all exist and satisfy the differential equations*

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial a}\right) = \sum_\mu f_{\mu\nu}(x; y_1, \cdots, y_n; a)\frac{\partial y_\mu}{\partial a} + f_{0\nu}(x; y_1, \cdots, y_n; a) \tag{4}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial a} = 0 \quad \text{for} \quad x = x_0. \tag{5}$$

We begin by proving the following lemma: *when, for* $x_0 \leqq x \leqq x_0 + h$, *the continuous function* $z = z(x)$ *satisfies the inequalities*

$$0 \leqq z \leqq \int_{x_0}^{x} (Mz + A)dx, \tag{6}$$

where the constants M and A are positive or zero, then

$$0 \leqq z \leqq Ahe^{Mh}, \qquad (x_0 \leqq x \leqq x_0 + h). \tag{7}$$

In analogy to the process of integrating a linear differential equation of the first order, we make $z = e^{M(x-x_0)} \cdot \zeta$; let the maximum of ζ on the closed interval $(x_0, x_0 + h)$ occur at $x = x_1$. For this value of x, (6) gives

$$0 \leqq e^{M(x_1-x_0)} \cdot \zeta_{\max} \leqq \int_{x_0}^{x_1} (Me^{M(x_1-x_0)}\zeta + A)dx,$$

whence by the first theorem of the mean

$$0 \leqq e^{M(x_1-x_0)}\zeta_{\max} \leqq \zeta_{\max} \int_{x_0}^{x_1} Me^{M(x-x_0)}dx + \int_{x_0}^{x_1} A dx$$
$$= \zeta_{\max}\,(e^{M(x_1-x_0)} - 1) + A(x_1 - x_0),$$

or finally

$$0 \leqq \zeta_{\max} \leqq A(x_1 - x_0) \leqq Ah,$$

from which (7) follows at once.

By (1) and (2) we have

$$y_\nu = y_\nu^0 + \int_{x_0}^{x} f_1(x;\, y_1, \cdots, y_n;\, a)dx;$$

denoting by y_ν' the solutions of (1) where the parameter a has been replaced by a' $(a_1 \leqq a' \leqq a_2)$ but the initial conditions remain unchanged, it follows that

$$y_\nu' - y_\nu = \int_{x_0}^{x} [f_\nu(x;\, y_1', \cdots, y_n';\, a) - f_\nu(x;\, y_1, \cdots, y_n;\, a)]dx$$

and consequently

$$\frac{y_\nu' - y_\nu}{a' - a} = \int_{x_0}^{x} \left[\sum_\mu \bar{f}_{\mu\nu} \cdot \frac{y_\mu' - y_\nu}{a' - a} + \bar{f}_{0\nu} \right] dx, \tag{8}$$

where $\bar{f}_{\mu\nu} = f_{\mu\nu}\ (x;\ \eta_{1\mu\nu}, \cdots, \eta_{n\mu\nu};\ \alpha_{\mu\nu})$, $\bar{f}_{0\nu} = f_{0\nu}\ (x;\ \eta_{10\nu}, \cdots, \eta_{n0\nu};\ \alpha_{0\nu})$ and the arguments $\eta_{\lambda\mu\nu}$, $\alpha_{\mu\nu}$ are intermediate between y_λ and y_λ', a and a' respectively.

Now denote by M a constant not less than any of the expressions $n|f_{\mu\nu}|$, and by A a constant not less than any of the expressions $|f_{0\nu}|$ for all values of $x, y_1, \cdots, y_n, a$ in the region (3). Moreover, denote by $z = z(x, a, a')$, where x is variable, the greatest of the n expressions

$$\left|\frac{y_\nu' - y_\nu}{a' - a}\right|;$$

for fixed values of a and a' ($\neq a$), this z is evidently a continuous function of x in $(x_0, x_0 + h)$. With these definitions of M, A and z, (8) immediately leads to (6), the lemma becomes applicable, and (7) gives

$$\left|\frac{y_\nu' - y_\nu}{a' - a}\right| \leqq Ahe^{Mh}. \tag{9}$$

In particular, it follows that the y_ν are uniformly continuous functions of a. By the existence theorem, the system of differential equation

$$\frac{dY_\nu}{dx} = \sum_\mu f_{\mu\nu}(x;\, y_1, \cdots, y_n;\, a)Y_\mu + f_{0\nu}(x;\, y_1, \cdots, y_n;\, a) \tag{4'}$$

with the initial conditions

$$Y_\nu = 0 \quad \text{for} \quad x = x_0 \tag{5'}$$

has a unique solution for x in $(x_0, x_0 + h)$ and a in (a_1, a_2). Our theorem is therefore proved if we show that

$$\frac{\partial y_\nu}{\partial a} = Y_\nu. \tag{10}$$

From (8), (4′) and (5′) it follows at once that

$$\begin{aligned} \frac{y_\nu' - y_\nu}{a' - a} - Y_\nu = \int_{x_0}^{x} \Bigg[\sum_\mu \bar{f}_{\mu\nu} \left(\frac{y_\mu' - y_\mu}{a' - a} - Y_\mu \right) \\ + \sum_\mu (\bar{f}_{\mu\nu} - f_{\mu\nu})Y_\mu + \bar{f}_{0\nu} - f_{0\nu} \Bigg] dx \end{aligned} \tag{11}$$

and on account of the continuity of y_ν in respect to a and the continuity of $f_{\mu\nu}, f_{0\nu}$ in their arguments, we may make each of the n expressions

$$\left|\sum_\mu (\bar{f}_{\mu\nu} - f_{\mu\nu})Y_\mu + \bar{f}_{0\nu} - f_{0\nu}\right|$$

less than any assigned ϵ by taking $|a' - a|$ sufficiently small. Denoting now by $z = z(x, a, a')$ the greatest of the n expressions

$$\left|\frac{y_\nu' - y_\nu}{a' - a} - Y_\nu\right|,$$

this z is a continuous function of x for fixed values of a and a', (11) leads to

(6) with $A = \epsilon$, and (7) gives

$$\left| \frac{y_\nu' - y_\nu}{a' - a} - Y_\nu \right| < \epsilon h e^{Mh}$$

uniformly for a, a' in (a_1, a_2), and this last inequality is equivalent to (10), which proves the theorem.

3. We now proceed to prove that *the partial derivatives $\partial y / \partial y_\lambda^0$ all exist and satisfy the differential equations*

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial y_\lambda^0}\right) = \sum_\mu f_{\mu\nu}(x;\, y_1, \cdots, y_n;\, a) \frac{\partial y_\mu}{\partial y_\lambda^0} \tag{12}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial y_\lambda^0} = \epsilon_{\lambda\nu} = \begin{cases} 1, & \lambda = \nu \\ 0, & \lambda \neq \nu \end{cases} \qquad \text{for} \qquad x = x_0. \tag{13}$$

This follows immediately from the preceding theorem by the substituting

$$y_\nu = z_\nu + y_\nu^0,$$

which gives the differential equations

$$\frac{dz_\nu}{dx} = f_\nu(x;\, z_1 + y_1^0, \cdots, z_n + y_n^0;\, a)$$

with the initial conditions $z_\nu = 0$ for $x = x_0$. Regarding y_λ^0 as the parameter in the preceding theorem, we find

$$\frac{d}{dx}\left(\frac{\partial z_\nu}{\partial y_\lambda^0}\right) = \sum_\mu f_{\mu\nu}\left(\frac{\partial z_\mu}{\partial y_\lambda^0} + \epsilon_{\lambda\mu}\right)$$

with the initial conditions $\partial z_\nu / \partial y_\lambda^0 = 0$ for $x = x_0$, whence (12) and (13).

4. Finally, supposing the continuity and existence conditions which hold in (3) to be fulfilled also in the extended region

$$x_0 - h_1 \leqq x \leqq x_0 + h, \qquad y_\nu^0 - k \leqq y_\nu \leqq y_\nu + k, \qquad a_1 \leqq a \leqq a_2,$$

the partial derivatives $\partial y_\nu / \partial x_0$ all exist and satisfy the differential equations

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial x_0}\right) = \sum_\mu f_{\mu\nu}(x;\, y_1, \cdots, y_n;\, a) \frac{\partial y_\mu}{\partial x_0} \tag{14}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial x_0} = -f_\nu(x_0;\, y_1^0, \cdots, y_n^0;\, a) \qquad \text{for} \qquad x = x_0. \tag{15}$$

Let y_ν' be the solutions of (1) when x_0 is replaced by x_0', all y_ν^0 remaining

the same; then we obtain in the same manner as (8) was found

$$\frac{y_\nu' - y_\nu}{x_0' - x_0} = \int_{x_0}^{x} \frac{f_\nu(x;\ y_1',\ \cdots,\ y_n';\ a) - f_\nu(x;\ y_1,\ \cdots,\ y_n;\ a)}{x_0' - x_0}\,dx$$

$$- \frac{1}{x_0' - x_0}\int_{x_0}^{x_0'} f_\nu(x;\ y_1',\ \cdots,\ y_n';\ a)dx$$

$$= \int_{x_0}^{x} \sum_\mu \bar{f}_{\mu\nu} \frac{y_\mu' - y_\mu}{x_0' - x_0}\,dx - \bar{f}_\nu.$$

Using the auxiliary system

$$\frac{dY_\nu}{dx} = \sum_\mu f_{\mu\nu} Y_\mu$$

with the initial conditions

$$Y_\nu = -f_\nu(x_0;\ y_1^0,\ \cdots,\ y_n^0;\ a) \qquad \text{for} \qquad x = x_0,$$

the proof then proceeds exactly as before.

ON QUATERNIONS AND THEIR GENERALIZATION AND THE HISTORY OF THE EIGHT-SQUARE THEOREM. ADDENDA.

By L. E. Dickson.

H. Scheffler* gave an erroneous 16-square formula

$$(\Sigma a_i^2)(\Sigma b_i^2) = \Sigma A_i^2,\ A_3 = a_8 b_6 - a_{15} b_{13} + \cdots,\ A_{12} = - a_8 b_{13} + a_{15} b_6 + \cdots,$$

whence ΣA_i^2 has the term $- 4a_8 a_{15} b_6 b_{13}$. The sign $+$ before $a_{15} b_{16}$ in A_2 is probably a misprint as it causes many cases of failures of the formula.

S. A. Corey communicated privately a special 16-square formula. In his identity† take $c_1 = c_2 = 1$ and let t_{2n-1} be conjugate to t_{2n} and r_{2n-1} conjugate to r_{2n}. Then $2S_8 S_8' = S_{16}$, where S_r denotes a sum of r squares. Evidently $2S_8 = S_8 + S_8 = S_{16}'$. As each of the numbers the sum of whose squares is S_8' take the hypotenuse of a right triangle, whence $S_8' = S_{16}''$. Thus $S_{16}' S_{16}'' = S_{16}$.

* Die Polydimensionalen Grössen und die Vollkommenen Primzahlen, Braunschweig, 1880, p. 94.

† Amer. Math. Monthly, 26, 1919, 72.

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